On ideals of \(n\)-homogeneous polynomials on Banach spaces

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Dedicated to Anastasios Mallios
on the occasion of his retirement

Abstract

This is mainly a survey on the basic ideas of a theory of quasi-normed ideals of \(n\)-homogeneous scalar-valued polynomials on Banach spaces. The ideas for this theory stem for a large part from Pietsch’s theory of operator ideals. As new results, a theorem on ultrastability will be extended from normed to \(\lambda\)-normed ideals and some ideals of polynomials factoring through \(L_r(\mu)\) will be studied.

1 Definitions and examples

1.1. The standard notation from Banach space theory will mostly be used. If \(E, E_1, \ldots, E_n, F\) are normed spaces over \(K = \mathbb{R}\) or \(\mathbb{C}\), then \(B_E\) denotes the closed unit ball, \(\mathcal{L}(E_1, \ldots, E_n; F)\) the space of continuous \(n\)-linear mappings, \(\mathcal{L}(nE; F) := \mathcal{L}(E, \ldots, E; F)\) (read \(n\)-times \(E\) for Nachbin’s notation \(n^E\)) and \(E' := \mathcal{L}(E; K)\). An \(n\)-homogeneous polynomial \(q : E \to K\) (in this paper only scalar-valued polynomials will be investigated) is a function such that there is an \(n\)-linear \(\varphi : E^n \to K\) with \(q(x) = \varphi(x, \ldots, x)\) for all \(x \in E\); there is a unique symmetric \(\varphi\) (denoted by \(\tilde{q}\)) with this property. \(\tilde{q}\) can be calculated with the polarization formula

\[ \tilde{q}(x_1, \ldots, x_n) = \frac{1}{n!} \int_0^1 r_1(t) \cdots r_n(t)q \left( x_0 + \sum_{k=1}^n r_k(t)x_k \right) dt \]

(where \(x_0, x_1, \ldots, x_n \in E\) and \(r_k : [0,1] \to \{-1,1\}\) are the Rademacher functions). \(q\) is continuous if and only \(\tilde{q}\) is; notation: \(q \in \mathcal{P}^n(E)\). With \(\|q\| := \sup\{|q(x)| \mid x \in B_E\}\) the space \(\mathcal{P}^n(E)\) is a Banach space. If \(x' \in E'\)
the function defined by $(\otimes^n x')(x) := (x', x)^n$ is an $n$-homogeneous polynomial. Polynomials of the form
\[ q = \sum_{k=1}^m \lambda_k \otimes^n x'_k \in P^n(E) \]
are called of finite rank; notation $q \in P^n_\infty(E)$. The elements in $P^n_\infty(E)$ (closure in $P^n(E)$) are called approximable.

1.2 A subclass $Q$ of the class $P^n$ of $n$-homogeneous polynomials on all Banach spaces (as is the case of operator ideals, it is common to consider only Banach and not all normed spaces) is called ideal, if the following three conditions are satisfied:

(a) The component $Q(E) := P^n(E) \cap Q$ is a linear subspace of $P^n(E)$ for each Banach space $E$.

(b) If $T \in \mathcal{L}(E; F)$ and $q \in Q(F)$, then $q \circ T \in Q$.

(c) $\otimes^n 1 = [K \ni z \mapsto z^n] \in Q$.

If $Q$ is an ideal and $Q : Q : [0, \infty[ \to N$, the pair $(Q, Q)$ is called a quasi-normed ideal of $n$-homogeneous (scalar-valued) polynomials if

(a′) $Q$ is a quasi-norm when restricted to $Q^n(E)$ (for all Banach spaces $E$)

(b′) $Q(q \circ T) \leq \|T\|^n Q(q)$ (in the situation of (b)).

(c′) $Q(\otimes^n 1 : K \to K) = 1$.

A simple argument with $\ell_p$-sums shows that the quasi-norm constants of $Q|_{Q(E)}$ can be chosen independently of $E$, say $c \geq 1$. The Aoki-Rolewicz result holds also in this situation: If $0 < \lambda \leq 1$ with $c = 2^{1-\lambda}$ one can define a $\lambda$-norm $Q_0$ by
\[ Q_0(q) := \inf \left\{ \left( \sum_{k=1}^m Q(q_k)^\lambda \right)^{\frac{1}{\lambda}} \mid m \in \mathbb{N}, q = \sum_{k=1}^m q_k \in Q(E) \right\} \]
satisfying $Q_0 \leq Q \leq 2^\lambda Q_0$ (see [P1, p. 92]). It is immediate to see, that $Q_0$ satisfies (b′) and (c′). The definiteness of $Q$ (i.e. $Q(q)$ implies $q = 0$) in
condition (a) holds automatically if the other conditions are satisfied. Easy arguments show that (b) and (c) imply

\[ \|q\| \leq Q(q) \quad \text{for all } q \in \mathcal{Q} \]

for all \( q' \in \mathcal{Q}(E) \) and \( Q(\otimes^n x') = \|x'||^n \) for all \( E \) and \( x' \in E' \).

In particular \( \mathcal{P}^n_f \subset \mathcal{Q} \) and \( \mathcal{Q}(M) = \mathcal{P}^n(M) \) if \( M \) is finite-dimensional. Note, that in this case all \( \mathcal{Q} \) generate in the finite-dimensional space \( \mathcal{P}^n(M) \) the same topology, the topology of pointwise convergence on \( M \).

1.3. \( \lambda \)-norms are continuous with respect to the topology they generate, quasi-norms in general not. To see this “quasi-norm catastrophe” for quasi-normed ideals consider the quasi-norm

\[ Q(q) := \begin{cases} \|q\| & \text{if } q \text{ is approximable} \\ 2\|q\| & \text{otherwise} \end{cases} \]

on \( \mathcal{P}^n \), then \( (\mathcal{P}^n, Q) \) is a quasi-normed ideal; if \( q_0 \notin \mathcal{P}_f^n(E) \) and \( 0 \neq q_1 \in \mathcal{P}_f(E) \), then \( \frac{1}{m} q_0 + q_1 \longrightarrow q_1 \) for \( m \to \infty \), but

\[ Q \left( \frac{1}{m} q_0 + q_1 \right) = 2\left\| \frac{1}{m} q_0 + q_1 \right\| \longrightarrow 2\|q_1\| \neq Q(q_1). \]

1.4. If all components \( Q(E) \) of a quasi-normed (resp. \( \lambda \)-normed) ideal are complete, \( (\mathcal{Q}, \mathcal{Q}) \) is called quasi-Banach, \( \lambda \)-Banach or Banach ideal respectively. The proof of the following result is standard.

**Criterion.** Let \( \mathcal{Q} \subset \mathcal{P}^n \) be a subclass, \( \mathcal{Q} : \mathcal{Q} \longrightarrow [0, \infty[ \) and \( 0 \leq \lambda \leq 1 \). Then \( (\mathcal{Q}, \mathcal{Q}) \) is a \( \lambda \)-Banach ideal if and only if

(a) If \( q_m \in \mathcal{Q}(E) \) with \( \sum_{m=1}^{\infty} Q(q_m)^\lambda < \infty \), then \( q := \sum_{m=1}^{\infty} q_m \) is well-defined (pointwise convergence), \( q \in \mathcal{Q} \) and \( Q(q)^\lambda \leq \sum_{m=1}^{\infty} Q(q_m)^\lambda \).

(b) If \( q \in \mathcal{Q}(F) \) and \( T \in \mathcal{L}(E; F) \), then \( q \circ T \in \mathcal{Q} \) and \( Q(q \circ T) \leq \|T\|^n Q(q) \).

(c) \( [\otimes^n 1 : \mathbb{K} \longrightarrow \mathbb{K}] \in \mathcal{Q} \) and \( Q(\otimes^n 1) = 1 \).

(d) \( Q(\lambda q) = |\lambda| Q(q) \) for all \( q \in \mathcal{Q} \) and \( \lambda \in \mathbb{K} \).
The last condition (d) follows from (b) if every \( \lambda \in K \) has an \( n \)-th root, i.e. \((K, n) \neq (\mathbb{R}, \text{even})\). This “bad” case \((\mathbb{R}, \text{even})\) is often exceptional in the theory. It is often useful to check first (b) and (c) since this implies \( \|q\| \leq Q(q) \) (see 1.2.) and hence the series in (a) converges in \( \mathcal{P}^n(E) \).

1.5. For a \( \lambda \)-normed ideal \((Q, Q)\) an \( n \)-homogeneous polynomial \( q \in \mathcal{P}^n(E) \) is in \( Q^{\max} \) if, by definition,

\[
Q^{\max}(q) := \sup \{ Q(q|_N) \mid N \in \text{FIN(E)} \} < \infty
\]

(where \( \text{FIN}(E) := \{ N \subset E \mid \text{finite-dimensional subspace} \} \)). It is easy to see that \((Q^{\max}, Q^{\max})\) is a \( \lambda \)-normed ideal. \((Q, Q)\) is maximal if \((Q, Q) = (Q^{\max}, Q^{\max})\). Obviously, \((Q^{\max}, Q^{\max})\) is maximal. The criterion 1.4. and the remark at the end of 1.2. about the convergence in \( \mathcal{P}^n(M) \) gives the Proposition.

Proposition. Let \((Q, Q)\) be a maximal \( \lambda \)-normed ideal of \( n \)-homogeneous polynomials and \( E \) a Banach space. Then the closed unit ball \( \{ q \in Q(E) \mid Q(q) \leq 1 \} \) of \( Q(E) \) is closed with respect to the pointwise topology on \( E \); in other words: If \( q_\lambda \in Q(E) \) such that \( c := \sup_{\lambda \in \Lambda} Q(q_\lambda) < \infty \) and \( q_\lambda \rightharpoonup q \) pointwise, then \( q \in Q(E) \) and \( Q(q) \leq c \). In particular: \( Q(E) \) is complete and \((Q, Q)\) a \( \lambda \)-Banach ideal.

1.6. Examples: (a) The classes \( \mathcal{P}^n, \mathcal{P}^n_f, \mathcal{P}^n_{f_f} \) of all, finite rank, approximable \( n \)-homogeneous polynomials are normed ideals with the usual norm.

(b) \( q : E \rightarrow K \) is called nuclear \((q \in N^n(E))\) if there are \( \lambda_m \in K \) and \( x'_m \in E' \) with

\[
\sum_{m=1}^{\infty} |\lambda_m| \|x'_m\|^n < \infty \quad \text{and} \quad q = \sum_{m=1}^{\infty} \lambda_m \otimes^n x'_m
\]

(the latter sum converges w.r.t. \( \|\| \)) and

\[
N^n(q) := \inf \left\{ \sum_{m=1}^{\infty} |\lambda_m| \|x'_m\|^n \mid q = \sum_{m=1}^{\infty} \lambda_m \otimes^n x'_m \right\}.
\]

This definition is due to Gupta [G]. It is straightforward that \((N^n, N^n)\) is a Banach ideal. Using \( Q(\otimes^n x') = \|x'\|^n \) for a normed ideal \((Q, Q)\) it is easy to see that \( N^n \subset Q \) for all Banach ideals and \( Q(q) \leq N^n(q) \): The ideal of \( n \)-homogeneous nuclear polynomials is the smallest Banach ideal. If \( \pi_n \) denotes the natural norm on the \( n \)-th symmetric tensor product \( \otimes^n E \) such
that \((\otimes^{n,s}E')^\prime = \mathcal{P}^n(E)\) (clearly, \(\overset{1}{=}\) means equality with equal norm), then the natural map coming from \(\otimes^nx \sim \otimes^nx\)

\[
\otimes^{n,s}E' \longrightarrow \mathcal{N}^n(E)
\]

is a metric surjection and \(\mathcal{N}^n(E) \overset{1}{=} \otimes^{n,s}E'\) if \(E'\) has the metric approximation property (see e.g. [F1] for symmetric tensor products of normed spaces). In particular: If \(E\) is reflexive and has the approximation property (whence \(E\) has the m.a.p.) one obtains \(\mathcal{N}^n(E') \overset{1}{=} \otimes^{n,s}E'\) and

\[
\mathcal{P}^n(E) \overset{1}{=} (\mathcal{N}^n(E'))';
\]

the duality bracket can be calculated for \(p \in \mathcal{P}^n(E)\) and \(q \in \mathcal{N}^n(E')\) by

\[
\langle p, q \rangle = \sum_{m=1}^{\infty} \lambda_{m} p(x_{m})
\]

for any nuclear representation \(q = \sum_{m=1}^{\infty} \lambda_{m} \otimes^nx_{m}\). This is the \textit{trace duality} for polynomials.

(c) Following Dineen [D1] an \(n\)-homogeneous polynomial \(q : E \longrightarrow \mathbb{K}\) is called \textit{integral} \((q \in \mathcal{I}^n(E))\) if there is a signed Borel-measure \(\mu\) on the \(\sigma(E',E)\)-compact \(B_{E'}\) such that

\[
q(x) = \int_{B_{E'}} \langle x', x \rangle^{n} \mu(dx') \quad \text{for all } x \in E;
\]

(*)

equivalently: if there is finite measure space \((\Omega, \mu)\) and \(T \in \mathcal{L}(E; L_{\infty}(|\mu|))\) such that

\[
q(x) = \int_{\Omega} [(Tx)(w)]^{n} \mu(dw)
\]

(**)

for all \(x \in E\). With

\[
\mathcal{I}^n(q) := \min \{ ||\mu|| \mid \text{with (*)} \} = \min \{ ||\mu|| ||T||^{n} \mid \text{with (**)} \}
\]

is a maximal normed ideal; if \((\mathbb{K}, n)\) is “good” (see 1.4.) the best measures can be chosen positive; moreover, it can be shown that \(\mathcal{I}^n(E) = (\otimes^{n,s}_{\epsilon}E')^{\prime}\) where \(\epsilon_{s}\) is the injective \(s\)-tensor norm (see again [F1] for details). It follows from the representation theorem of maximal ideals by an \(s\)-tensor norm \(\alpha\) (see [FH]) and \(\epsilon_{s} \leq \alpha\) that \(\mathcal{I}^n\) is the smallest \textit{maximal} normed ideal. Note that \(\mathcal{I}^n(E) \overset{1}{=} \mathcal{N}^n(E)\) if \(E'\) has the Radon-Nikodym property – a result of
Alencar [A], Boyd-Ryan [BR] and Carando-Dimant [CD] (see also [F1]). In particular, \( N^n(M) \parallel I^n(M) \) for all finite-dimensional \( M \); this implies that \( (N^n)_{\text{max}} \parallel I^n \).

(d) There are many other interesting ideals of \( n \)-homogeneous polynomials which were studied in the literature – mostly vector-valued where, clearly, new interesting phenomena occur. I just mention those polynomials which are somehow “summing” (e.g.: that they through weakly \( r \)-summable sequences in “better” summable ones); see e.g. the papers of Alencar, Botelho, Braunss, Junek, Matos, and Waltz which are cited in the list of references.

1.7. The closed graph theorem and an \( \ell_r \)-sum argument give the

**Proposition.** If \((Q_j, Q_j)\) (for \( j = 1, 2 \)) are quasi-Banach ideals of \( n \)-homogeneous polynomials with \( Q_1 \subset Q_2 \), then there is a \( c > 0 \) with \( Q_2 \leq cQ_1 \).

1.8. It is shown in [F2] that for every maximal normed ideal \((Q, Q)\) of \( n \)-homogeneous polynomials there is a maximal normed ideal \((A, A)\) of \( n \)-linear functionals on Banach spaces such that \( q \in Q \) if and only if \( \tilde{q} \in A \); moreover there is a constant \( c > 0 \) with

\[
\frac{1}{c} A(\tilde{q}) \leq Q(q) \leq c A(\tilde{q})
\]

but \( c \neq 1 \) in general. It follows that the isomorphic theory of maximal normed ideals of \( n \)-homogeneous polynomials can be treated with ideals of \( n \)-linear functionals.

1.9. I was not able to localize where the notion of an ideal of \( n \)-homogeneous polynomials was first used. It was certainly in the air in the late sixties after Pietsch had started studying systematically operator ideals and Nachbin had introduced holomorphy types; the latter is (in the scalar-valued case) a sequence \((P^n_\theta, \| \cdot \|_{\theta, n})_{n=0,1,{\ldots}}\) of subclasses \( P^n_\theta \subset P^n \) (for \( n \geq 1 \)) with all components \( P^n_\theta(E) \) being linear subspaces of \( P^n(E) \), normed with \( \| \cdot \|_{\theta, n} \) and \( P^0_\theta(E) \parallel \mathbb{C} \) such that a certain quantitative stability under differentiation holds true (see [N]). In the special examples (e.g. all continuous, nuclear, integral, approximable or Hilbert-Schmidt) which were investigated, the \( P^n_\theta \) were ideals. When Pietsch [P2] marked in 1983 the beginning of the study of ideals of \( n \)-linear maps it seems to me that it became somehow folklore that there is a notion of ideals of \( n \)-homogeneous polynomials.
2 Ultrastability

2.1. Let \( \mathcal{U} \) be an ultrafilter on a set \( I \) and \( E_i \) a Banach space for each \( i \in I \). If

\[
\ell_{\infty}(I; E_i) := \{(x_i) \in \ell_{\infty}(I; E_i) \mid \lim_{i, \mathcal{U}} \|x_i\|_{E_i} = 0\}
\]

then the ultraproduct \((E_i)_\mathcal{U}\) is defined to be

\[
\ell_{\infty}(I; E_i) / c_0(I; E_i)
\]

with the quotient norm \(\|(x_i)_\mathcal{U}\| = \lim_{i, \mathcal{U}} \|x_i\|_{E_i}\). If all \( E_i = E \), then \( J_E : E \overset{1}{\rightarrow} (E)_\mathcal{U} \) (via \( x \mapsto (x)_\mathcal{U} \)) where \( \overset{1}{\rightarrow} \) means “isometry into”; if \( \dim E < \infty \) this map is onto, in particular \((\mathbb{K})_\mathcal{U} = \mathbb{K}\).

The following result will be needed, the proof of which is a slight modification of [H, 6.1].

**Local determination of ultraproducts.** If \( M \in \text{FIN}((E_i)_\mathcal{U}) \), then there are \( R_i \in \mathcal{L}(M; E_i) \) for all \( i \in I \) with \( \|R_i\| \leq 1 \) and \( x = (R_i x)_\mathcal{U} \) for all \( x \in M \).

If \( T_i \in \mathcal{L}(E_i^1, \ldots, E_i^n; F_i) \) with \( \sup_{i \in I} \|T_i\| < \infty \), then

\[
(T_i)_\mathcal{U}((x_i^1)_\mathcal{U}, \ldots, (x_i^n)_\mathcal{U}) := (T_i(x_i^1, \ldots, x_i^n))_\mathcal{U}
\]

defines an \( n \)-linear map: \( (E_i^1)_\mathcal{U} \times \cdots \times (E_i^n)_\mathcal{U} \rightarrow (F_i)_\mathcal{U} \) with \( \|(T_i)_\mathcal{U}\| = \lim_{i, \mathcal{U}} \|T_i\| \). In the case of \( F_i = \mathbb{K} \) for all \( i \in I \) one obtains that \((T_i)_\mathcal{U}\) has range in \((\mathbb{K})_\mathcal{U} = \mathbb{K}\) and

\[
(T_i)_\mathcal{U}((x_i^1)_\mathcal{U}, \ldots, (x_i^n)_\mathcal{U}) = \lim_{i, \mathcal{U}} T_i(x_i^1, \ldots, x_i^n).
\]

In particular: If \( q_i \in \mathcal{P}(E_i) \) with \( \sup_{i \in I} \|q_i\| < \infty \), then

\[
[\lim_{i, \mathcal{U}} q_i](x_i)_\mathcal{U} := \langle q_i \rangle\mathcal{U}((x_i)_\mathcal{U}, \ldots, (x_i)_\mathcal{U}) = \lim_{i, \mathcal{U}} q_i(x_i)
\]

is a well-defined continuous \( n \)-homogeneous polynomial. It is straightforward to see that also \( \|\lim_{i, \mathcal{U}} q_i\| = \lim_{i, \mathcal{U}} \|q_i\| \).

2.2. A \( \lambda \)-normed ideal \((Q, Q)\) of \( n \)-homogeneous polynomials is called **ultrastable** if for every ultrafilter the following holds: if \( q_i \in Q(E_i) \) and \( \sup_{i \in I} Q(q_i) < \infty \), then

\[
\lim_{i, \mathcal{U}} q_i \in Q \quad \text{and} \quad Q(\lim_{i, \mathcal{U}} q_i) \leq \lim_{i, \mathcal{U}} Q(q_i).
\]
It is clear that the condition on $Q$ is equivalent to:

$$Q\left(\lim_{\lambda\in\mathcal{U}} q\right) \leq 1 \quad \text{if all } Q(q_\lambda) \leq 1.$$ 

**Theorem.** For every $\lambda$-normed ideal $(Q, \mathcal{Q})$ of $n$-homogeneous polynomials the following are equivalent:

(a) $(Q, \mathcal{Q})$ is maximal.

(b) $(Q, \mathcal{Q})$ is ultrastable.

This is a sort of generalization of the Kürsten-Heinrich theorem that an operator ideal is maximal if and only if it is ultrastable (see [HI]); the result for normed ideals of polynomials was shown in [FH] with the help of the already mentioned representation theorem with $s$-tensor norms. The following proof uses Heinrich’s ideas in [H] for the linear case.

**Proof.** Assume first that $(Q, \mathcal{Q})$ is ultrastable and let $q \in Q^{\text{max}}$. Take $\mathcal{U}$ an ultrafilter on $\text{FIN}(E)$ which is finer than the order filter and consider

$$f_M : E \to M \quad \text{with} \quad f_M(x) := \begin{cases} x & \text{if } x \in M \\ 0 & \text{otherwise,} \end{cases}$$

then $\Phi : E \to (M)_{\mathcal{U}}$ defined by $\Phi(x) = (f_M(x))_{\mathcal{U}}$ is an isometry. It follows that

$$q(x) = \lim_{M,\mathcal{U}} q(f_M(x)) = \left[\lim_{M,\mathcal{U}} q\right](\Phi(x))$$

hence $q = \left[\lim_{M,\mathcal{U}} q\right] \circ \Phi \in Q$ by ultrastability (and the ideal property) and

$$Q(q) \leq \|\Phi\| \cdot Q\left(\lim_{M,\mathcal{U}} q\right) \leq \lim_{M,\mathcal{U}} Q(q) \leq Q^{\text{max}}(q).$$

Since always $Q^{\text{max}} \leq Q$ it follows that $Q(q) = Q^{\text{max}}(q)$.

Vice versa, assume that $(Q, \mathcal{Q})$ is maximal, take an ultrafilter $\mathcal{U}$ on a set $I$ and $q_\lambda \in Q(E_\lambda)$ with $Q(q_\lambda) \leq 1$ for all $\lambda \in I$ and set $q := \lim_{\lambda,\mathcal{U}} q_\lambda$. By the maximality it is enough to show that $Q(q_M) \leq 1$ for all $M \in \text{FIN}((E_\lambda)_{\mathcal{U}})$. For such an $M$ choose (according to the local determination of ultraproduts) $R_\lambda \in \mathcal{L}(M; E_\lambda)$ with $\|R_\lambda\| \leq 1$ and $x = (R_\lambda x)_{\mathcal{U}}$ for all $x \in M$. It follows that

$$q(x) = q((R_\lambda x)_{\mathcal{U}}) = \lim_{\lambda,\mathcal{U}} q_\lambda(R_\lambda x)$$
for all $x \in M$. Hence $q|_M$ is the pointwise limit of $(q_i \circ R_i)_{i \in I}$ along the ultrafilter $\mathcal{U}$ in the finite-dimensional space $\mathcal{P}^n(M)$ and therefore also with respect to the topology generated by $Q$ on $\mathcal{P}^n(M)$. Since $Q$ is continuous (it is a $\lambda$-norm!) and $Q(q_i \circ R_i) \leq Q(q_i) \| R_i \|^n \leq 1$ it follows that $Q(q|_M) \leq 1$ for all $M \subset \text{FIN}((E_i)_\mathcal{U})$.

It is somehow interesting to observe that the proof showed that ultra-stability only for finite-dimensional $E_i$ implies the “general” ultrastability.

2.3. Just one application of this result: Using the principle of local reflexivity one can find, for each Banach space $E$, an ultrafilter $\mathcal{U}$ such that there is an isometry (into)

$$\Phi : E'' \longrightarrow (E)_{\mathcal{U}}$$

with $\Phi(x) = (x)_{\mathcal{U}}$ for all $x \in E$ (see e.g. [F1, 6.1] for details). If $q \in \mathcal{P}^n(E)$, then $\overline{q}_{\mathcal{U}} [(x)_{\mathcal{U}}] := \lim_{\mathcal{U}} q(x_i)$ is defined on $(E)_{\mathcal{U}}$ and $\overline{q}_{\mathcal{U}} := \overline{q}_{\mathcal{U}} \circ \Phi \in \mathcal{P}^n(E'')$ extends $q$ to $E''$; clearly $\|q\| = \|\overline{q}_{\mathcal{U}}\|$. This extension is called an uniterated Aron-Berner extension of $q$ (along $\mathcal{U}$) in [F1]. Now it is immediate that

**Corollary.** If $(\mathcal{Q}, Q)$ is a maximal $\lambda$-normed ideal, then $q \in \mathcal{Q}(E)$ if and only if $\overline{q}_{\mathcal{U}} \in \mathcal{Q}(E'')$; in this case $Q(q) = Q(\overline{q}_{\mathcal{U}})$.

3 The ideal of $r$-factorable $n$-homogeneous polynomials

3.1. Let $1 \leq r \leq \infty$. An $n$-homogeneous polynomial is $q : E \longrightarrow \mathbb{K}$ called $r$-factorable if there is a positive measure space $(\Omega, \mu)$, an operator $T \in \mathcal{L}(E; L_r(\Omega, \mu))$ and $p \in \mathcal{P}^n(L_r(\Omega, \mu))$ with $q = p \circ T$; notation: $q \in \mathcal{L}_r^n(E)$. Define

$$L_r^n(q) := \inf \left\{ \|T\|^{\frac{n}{r}} \|p\| \mid q : E \xrightarrow{T} L_r(\mu) \xrightarrow{p} \mathbb{K} \right\}$$

Note that $L_r^n(q : L_r(\mu) \longrightarrow \mathbb{K}) = \|q\|$. 

**Proposition.** $(\mathcal{L}_r^n, L_r^n)$ is a maximal, ultrastable $\lambda$-Banach ideal with $\lambda := \frac{r}{n}$ if $r \leq n$ and $\lambda = 1$ if $r \geq n$. 9
Proof. (a) \( L^n_r(\otimes^n 1 : \mathbb{K} \to \mathbb{K}) = \|\otimes^n 1\| = 1 \), the ideal property and the homogeneity of \( L^n_r \) are straightforward. For the triangle property assume first \( r \leq n \) and, for \( q_j \in L^n_r(E) \), choose factorizations
\[
q_j : E \xrightarrow{T_j} L_r(\Omega_j, \mu_j) \xrightarrow{p_j} \mathbb{K} \quad \text{for} \ j = 1, 2
\]
with \( \|p_j\| = 1 \) and \( \|T_j\|^n \leq L^n_r(q_j) + \varepsilon \). Define \( \Omega := \Omega_1 \cup \Omega_2 \) and \( \mu_1 \oplus \mu_2 \) on \( \Omega \)
\[
T x := T_1 x + T_2 x \in L_r(\Omega, \mu_1 \oplus \mu_2) = L_r(\Omega_1, \mu_1) \oplus_r L_r(\Omega_2, \mu_2)
\]
\[
p(f_1 \oplus f_2) := p_1(f_1) + p_2(f_2)
\]
then it is straightforward that \( q_1 + q_2 = p \circ T \) and
\[
\|T\| \leq (\|T_1\|^r + \|T_2\|^r)^{\frac{1}{r}}
\]
\[
|p(f_1 \oplus f_2)| \leq \|p_1\| \|f_1\|^n + \|p_2\| \|f_2\|^n \leq \|f_1\|^n + \|f_2\|^n \leq \left( \|f_1\|^r + \|f_2\|^r \right)^{\frac{n}{r}} = \|f_1 \oplus f_2\|^n
\]
hence \( \|p\| \leq 1 \) and
\[
L^n_r(q_1 + q_2) \leq \|T\|^n \|p\| \leq \left( (L^n_r(q_1) + \varepsilon)^{\frac{n}{r}} + (L^n_r(q_2) + \varepsilon)^{\frac{n}{r}} \right)^{\frac{n}{r}}.
\]
This gives the desired inequality and \( L^n_r \) is an \( \frac{n}{r} \)-norm (for definiteness recall a remark in 1.2.).

(b) If \( n \leq r < \infty \) the proof of the triangle inequality uses a trick known from Lapresté’s tensor norms: Take \( \frac{n}{r} + \frac{1}{s} = 1 \) and factorizations of \( q_1 \) and \( q_2 \) as before but this time with
\[
\|T_j\|^r = \|p_j\|^s \leq L^n_r(q_j) + \varepsilon.
\]
With \( T \) and \( p \) as in (a) one obtains
\[
\|T\| \leq \left( \|T_1\|^r + \|T_2\|^r \right)^{\frac{1}{r}} \leq (L^n_r(q_1) + L^n_r(q_2) + 2\varepsilon)^{\frac{1}{r}}
\]
\[
|p(f_1 \oplus f_2)| \leq \|p_1\| \|f_1\|^n + \|p_2\| \|f_2\|^n \leq \left( \|p_1\|^s + \|p_2\|^s \right)^{\frac{1}{s}} \left( \|f_1\|^{ns'} + \|f_2\|^{ns'} \right)^{\frac{1}{s'}} \leq (L^n_r(q_1) + L^n_r(q_2) + 2\varepsilon)^{\frac{1}{s}} (\|f_1 \oplus f_2\|)^n
\]
obtains that \( \lim F \) space, hence with \( \| q \| \) and \( \| q \| \) e.g. 2.3.), hence \( L \) some positive measure if one has to use the bidual, as above.

and \( L \) representation theorems of Kakutani-Bohnenblust-Nakano, [DF, p. 227]) that the ultraproduct of abstract \( L \) 3.2.

An immediate consequence of 2.3. (and the proposition) is that for every \( q \in L \) with \( \left\| \lim_{\mathcal{U}} p_i \right\| \leq 1 \), it is rather easy to see (e.g. [DF, p. 227]) that the ultraproduct of abstract \( L \)-spaces is an abstract \( L \)-space, hence \( F := (L_r(\mu))_{\mathcal{U}} \) is an abstract \( L \)-space and, therefore, by the representation theorems of Kakutani-Bohnenblust-Nakano, \( F = L_r(\mu) \) for some positive measure if \( r < \infty \). It follows \( \lim_{\mathcal{U}} q_i \in L^n_\mathcal{U} \) and \( L^n_\mathcal{U}(\lim_{\mathcal{U}} q_i) \leq \left\| (T_i)_{\mathcal{U}} \right\|^{n} \| \lim_{\mathcal{U}} p_i \| \leq 1 \). If \( r = \infty \) (the case of abstract \( M \)-spaces) one obtains that \( F'' = L_{\infty}(\mu) \) for some positive measure \( \mu \). The polynomial \( \lim_{\mathcal{U}} p_i \) on \( F \) has an extension \( p \) to \( F'' = L_{\infty}(\mu) \) with \( \| p \| = \| \lim_{\mathcal{U}} p_i \| \) (see e.g. 2.3.), hence \( q = p \circ \kappa_F \circ (T_i)_{\mathcal{U}} \) and \( L^n_\infty(q) \leq 1 \) as well. \( \square \)

It is likely that \( L^n_\mathcal{U} \) is not normed for \( n > r \); it would be interesting to see an example for \( L^n_3 \), the polynomials which factor through a Hilbert space.

An immediate consequence of 2.3. (and the proposition) is that for every \( q \in L^n(E) \) the uniterated Aron-Berner extensions \( T^n \) are in \( L^n(E''') \) with same \( L^n_\mathcal{U} \)-norm.

3.2. The ultrastability argument can be used to show that every \( q \in L^n_\mathcal{U} \) has a factorization \( q = p \circ T \) with

\[
L^n_r(q) = \| T \|^{n} \| p \|
\]

(the infimum is attained): Take an ultrafilter \( \mathcal{U} \) on \( [0,1] \) which is finer than the decreasing order filter and factorizations

\[
q : E \xrightarrow{T_\varepsilon} L_r(\mu_\varepsilon) \xrightarrow{p_\varepsilon} K
\]

with \( \| T_\varepsilon \| = 1 \) and \( \| p_\varepsilon \| \leq L^n_r(q) + \varepsilon \). Now \( q \) has a factorization (for \( r < \infty \))

\[
q : E \xrightarrow{J_E} (E)_{\mathcal{U}} \xrightarrow{(T_\varepsilon)_{\mathcal{U}}} (L_r(\mu_\varepsilon))_{\mathcal{U}} = L_r(\mu) \xrightarrow{\lim_{\mathcal{U}} p_\varepsilon} K
\]

and \( L^n_r(q) \leq \| (T_i)_{\mathcal{U}} \circ J_E \| \left\| \lim_{\mathcal{U}} p_\varepsilon \right\| \leq 1 \cdot \left\| \lim_{\mathcal{U}} p_\varepsilon \right\| = L^n_r(q) \). For \( r = \infty \) one has to use the bidual, as above.
3.3. For \( n = 2 \), the theory of operator ideals can be applied; for example Grothendieck’s inequality (in the form that every operator \( \ell_\infty \to \ell_1 \) is \( 2 \)-dominated, [DF, 17.14.]) and Kwapien’s factorization theorem give \( \mathcal{P}^2(E) = \mathcal{L}^2(E) \) for all \( \mathcal{L}^2_\infty \)-spaces (see [DF, 23.] and \( \mathcal{L}^2_\infty(q) \leq \frac{9}{1} 3 \cdot K_G \lambda^2 \|q\| \leq 7\lambda^2 \|q\| \)). I omit the details (in 4.4. a similar calculation will be made). Note that \( C(K) \) and \( L_\infty \) are \( \mathcal{L}^2_\infty,1 \)-spaces.

3.4. The case \( r = \infty \) is well-known (see Kirwan-Ryan [KR]):

**Proposition.** For each Banach space \( E \) and \( q \in \mathcal{P}^n(E) \) the following are equivalent:

1. \( q \) is \( \infty \)-factorable.
2. \( q \) has a factorization \( E \xrightarrow{T} C(K) \xrightarrow{p} \mathbb{K} \) for some compact \( K \).
3. \( q \) is extendible, i.e. for each superspace \( F \supset E \) there is an extension \( \tilde{q} \in \mathcal{P}^n(F) \) of \( q \).

**Proof.** (1) \( \Rightarrow \) (3): Take a factorization \( q : E \xrightarrow{T} L_\infty(\mu) \xrightarrow{p} \mathbb{K} \) with \( \|T\| = 1 \) and \( \|p\| = \mathcal{L}^n_\mu(q) \) (see 3.2.) and \( F \supset E \). Since \( L_\infty \) has the metric extension property there is an extension \( \tilde{T} \in \mathcal{L}(F; L_\infty(\mu)) \) with \( \|\tilde{q}\| \leq \|T\| \|p\| = \mathcal{L}^n_\mu(q) \).

(3) \( \Rightarrow \) (2) holds since every Banach space is isometrically contained in a \( C(K) \).

(2) \( \Rightarrow \) (1) follows using the uniterated Aron-Berner extension \( \tilde{q}^d \) of \( q \) to the bidual \( C(K)^\prime\prime \) (see 2.3.) which is an \( L_\infty \).

3.5. An operator \( T \in \mathcal{L}(E; F) \) is called \( r \)-factorable \( (1 \leq r \leq \infty) \) if there are a positive measure \( \mu \) and linear continuous \( R, S \) such that

\[
\begin{array}{ccc}
E & \xrightarrow{T} & F'' \\
R \downarrow & & \downarrow S \\
L_r(\mu) & \xleftarrow{sp} & F''
\end{array}
\]

holds. With \( \mathcal{L}_r(T) := \inf\{\|R\| \|S\| \mid \text{as in (*)} \} \) one obtains a maximal Banach operator ideal \( (\mathcal{L}_r, \mathcal{L}_r) \) (see e.g. [DF]). It is straightforward to show that \( q \in \mathcal{P}^n(E) \) is \( r \)-factorable if and only if it has a factorization \( q = p \circ T \) with \( T \in \mathcal{L}_r(E; F) \) and \( p \in \mathcal{P}^n(F) \); in this case \( \mathcal{L}_r^n(q) := \inf\{\mathcal{L}_r(T)^n \|q\| \mid q = p \circ T\} \) holds.
3.6. Therefore $L^n_r$ is a special case of the following general construction: Let $(Q, Q)$ be an $s$-normed ideal of $n$-homogeneous polynomials and $(A, A)$ a $t$-normed operator ideal. Then $q \in P^n(E)$ is defined to be in $Q_A(E)$ if it has a factorization

$$q : E \xrightarrow{T \in A} F \xrightarrow{p \in Q} K.$$

Define $Q_A(q) := \inf \{ A(T)^n Q(p) \mid q = p \circ T \}$ and $0 < \lambda < 1$ with $\frac{t}{s} + \frac{1}{s} = \frac{1}{\lambda}$.

**Proposition.**

(a) $(Q_A, Q_A)$ is a $\lambda$-normed ideal of $n$-homogeneous polynomials.

(b) It is complete, if $(Q, Q)$ and $(A, A)$ are complete.

(c) If $(Q, Q)$ and $(A, A)$ are maximal, then $(Q_A, Q_A)$ is a maximal, ultrastable $\lambda$-Banach ideal.

**Proof.** (a) The only not-immediate property is the additivity: Take $q_j \in Q_A(E)$ with factorizations $E \xrightarrow{T_j} F_j \xrightarrow{p_j} K$ satisfying

$$A(T_j)^{t} = Q(p_j)^{s} \leq Q_A(q_j)^{\lambda} + \varepsilon$$

with $T := T_1 \oplus T_2 : E \to F_1 \oplus F_2$ and $p : F_1 \oplus F_2 \to K$ with $p(x_1 \oplus x_2) := p_1(x_1) + p_2(x_2)$ one obtains with the ideal properties that

$$A(T)^{t} \leq A(T_1)^{t} + A(T_2)^{t} \quad \text{and} \quad Q(p)^{s} \leq Q(p_1)^{s} + Q(p_2)^{s}$$

which implies that $q_1 + q_1 = p \circ T$ is in $Q_A$ and the $\lambda$-triangle inequality.

(b) Using the criterion 1.4. and $\ell_1(F_j)$ the same trick gives the completeness.

(c) If $A$ is maximal, it is ultrastable by the Kürsten-Heinrich theorem (see [H]), $Q$ is ultrastable by 2.2. To see the ultrastability of $Q_A$ (which implies maximality and completeness), take an ultrafilter $\mathcal{U}$ on $I$ and $q_i \in Q_{\mathcal{U}}(E_i)$ with $Q_A(q_i) < 1$ and factorizations $q_i = p_i \circ T$ with $A(T_i : E \to F_i) \leq 1$ $Q(p_i) \leq 1$. Then

$$\lim_{i, \mathcal{U}} q_i : (E_i)_{\mathcal{U}} \xrightarrow{(T_i)_{\mathcal{U}}} (F_i)_{\mathcal{U}} \xrightarrow{\lim_{i, \mathcal{U}} p_i} K$$

gives the desired factorization. \(\square\)
This type of creating ideals is related to a construction of Geiss [G] for ideals of \( n \)-linear mappings.

3.7. Note that though \( \lambda \) is \( < 1 \), the ideal may be normed: Take \( L^r_n \) with \( r \geq n \) as an example. But the following example shows that \( Q_A \) need not be normed, if \( Q \) and \( A \) are: Take \( Q := I^2 \) the ideal of 2-homogeneous integral polynomials and \( A := I \) the integral operators. If \( T^2_n \) were Banach it would contain \( N^2 \) (see 1.6.(b)), in particular: \( q := \sum_{m=1}^{\infty} \lambda_m \otimes e_m \in T^2_n(\ell_2) \) for all \( (\lambda_m) \in \ell_1 \). Since integral operators on \( \ell_2 \) factor through a Hilbert-Schmidt operator \( T \in L(\ell_2; H) \) and \( p \in T^2(\mu) = N^2(\mu) \) (see 1.6.(c)), it follows that there is a nuclear \( S : H \to H' \) with \( U := \sum_{m=1}^{\infty} \lambda_m e_m \otimes e_m = T' ST \in L(\ell_2; \ell_2') \), hence \( U \in S_2 \circ S_1 \circ S_2 \subset S_{1/2} \) (Schatten classes, see e.g. [P1, 15.5.]) and therefore \( (\lambda_n) \in \ell_{1/2} \) for all \( (\lambda_n) \in \ell_1 \). A contradiction. It follows that \( T^2_n \) is not normed.

4 Related ideals

4.1. In this section two ideals related to the ideal \( L^r_n \) of \( r \)-factorable polynomials are studied: In the factorization \( q : E \to L^r(\mu) \xrightarrow{p} K \) the polynomial \( p \) should be not only continuous, but also positive or even an integrating functional. The results have a preliminary character. I always assume in this section the “good” case for \( (K, n) \), i.e. the cases \((\mathbb{R}, \text{even})\) are excluded and every \( \lambda \in K \) has an \( n \)-th root.

4.2. An \( n \)-homogeneous polynomial \( p : F \to K \) on a Banach lattice \( E \) is called positive, if \( \bar{p} : F^n \to K \) is positive, i.e. \( \bar{p}(f_1, \ldots, f_n) \geq 0 \) if all \( f_j \geq 0 \). A polynomial \( q \in \mathcal{P}^n(E) \) is called positively \( r \)-factorable \( (1 \leq r \leq \infty) \) if it admits a factorization

\[
q : E \xrightarrow{T} L^r(\mu) \xrightarrow{p} K
\]

with a positive measure \( \mu \), an operator \( T \in \mathcal{L}(E; L^r(\mu)) \) and \( p \in \mathcal{P}^n(L^r(\mu)) \) being positive; notation: \( q \in \mathcal{J}^n_r(E) \). Clearly \( \mathcal{J}^n_r \subset \mathcal{L}^n_r \). Define

\[
\mathcal{J}^n_r(q) := \inf \{ \| T \|^n \| p \| \mid q = p \circ T \text{ as before} \}.
\]

hence \( L^n_r \leq \mathcal{J}^n_r \). Since it is easily seen that the limit of \( p_i \) is positive if all \( p_i \) are (and the limit exists), exactly the same proof as for \( L^n_r \) gives the

**Proposition.** \( (\mathcal{J}^n_r, \mathcal{J}^n_r) \) is a maximal, ultrastable \( \lambda \)-Banach ideal with \( \lambda := \min\{1, \frac{r}{n}\} \). Moreover, the infimum in the definition of \( \mathcal{J}^n_r \) is attained.
Note, that the proof of the homogeneity of $J^n_r(E)$ requires that $(K,n)$ is good!

**4.3.** If $\mu$ is a finite, positive measure on $\Omega$ and $n \leq r \leq \infty$, the $n$-th integrating polynomial $q_{\mu,r}^n : L_r(\mu) \to K$ is defined by $q_{\mu,r}^n(f) := \int f^n d\mu$. It is straightforward to see that $\|q_{\mu,r}^n\| = (\mu(\Omega))^{\frac{n}{n}}$ where $s = n \frac{1}{r}$. A polynomial $q \in P^n(E)$ is called $r$-integral if it admits a factorization

$$q : E \xrightarrow{T} L_r(\mu) \xrightarrow{q_{\mu,r}^n} K$$

with a finite, positive measure $\mu$ and $T \in \mathcal{L}(E; L_r(\mu))$; notation: $q \in I^n_r(E)$ and

$$I^n_r(E) := \{\|T\| \|q_{\mu,r}^n\| : q = q_{\mu,r}^n \circ T \text{ as before}\}.$$

The name “$r$-integral” which I propose for these polynomials may be mis-leading in the case $n = 2$, since the linear operator associated with $(q_{\mu,r}^2)^{\vee} \in \mathcal{L}(L_r(\mu))$ is just the embedding $L_r(\mu) \hookrightarrow L_r(\mu)$ which is the typical $(r', r')$-factorable linear operator and not the typical $r$-integral operator (which is $L_\infty \hookrightarrow L_r$). Recall that in the linear theory $r$-factorable $= (r, r')$-factorable (see [DF, 18.] for details).

**Proposition.**

(a) $(I^n_r, I^n_\infty)$ is a Banach ideal of $n$-homogeneous polynomials.

(b) $I^n_r \subset J^n_r \subset L^n_r$ and $L^n_r \leq J^n_r \leq I^n_r$

(c) $(I^n_\infty, I^n_\infty) = (I^n_r, I^n_\infty)$ the integral $n$-homogeneous polynomials.

**Proof.** The ideal property (b) and $I^n_r(\otimes^{n}1) = 1$ of the criterion 1.4. are immediate. Now take $q_m \in I^n_r(E)$ with $c := \sum_{m=1}^{\infty} I^n_r(q_m) < \infty$, define $q := \sum_{m=1}^{\infty} q_m \in P^n(E)$ (see the remark at the end of 1.4.) and factor

$$q_m : E \xrightarrow{T_m} L_r(\Omega_m, \mu_m) \xrightarrow{q_{\mu_m,r}^n} K$$

with $\mu_m(\Omega_m) = \|T_m\| \leq I^n_r(q_m)(1 + \varepsilon)$. Then $\mu := \otimes_{m=1}^{\infty} \mu_m$ on $\Omega := \bigcup \{\Omega_m \mid m \in \mathbb{N}\}$ has measure $\leq c(1 + \varepsilon)$ and $T := \sum_{m=1}^{\infty} T_m : E \to L_r(\Omega, \mu)$ defined by $\otimes_{m=1}^{\infty} T_m x$ has norm $\leq (\sum_{m=1}^{\infty} \|T_m\|)^{\frac{1}{2}}$. Since

$$q_{\mu,m}^n(Tx) = \int_{\Omega_m} \left( \sum_{m=1}^{\infty} T_m x(w) \right)^n \mu_m(dw) = \sum_{m=1}^{\infty} \int_{\Omega_m} [T_m x(w)]^n \mu_m(dw) = \sum_{m=1}^{\infty} q_m(x) = q(x)$$

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it follows that
\[ I^n_r(q) \leq \|T\|^n \mu(\Omega)^{1/2} \left( \sum_{m=1}^{\infty} \|T_m\|^r \right)^{\theta} (c(1+\varepsilon))^{1/2} \leq [c(1+\varepsilon)]^{\theta + 1/2} = c(1+\varepsilon) \]
which was to be shown. Statement (b) is trivial and (c) follows from 1.6.(c).

4.4. In the case \( n = 2 \) (and \( 2 \leq r \leq \infty \)) the two last ideals coincide:

**Proposition.** \( J^2_r = I^2_r \) and \( J^2_r \leq I^2_r \leq c_r J^2_r \) for some \( c_r > 0 \).

**Proof.** It is enough to show that every positive \( q : L_r(\mu) \to \mathbb{K} \) is in \( I^2_r \) since this implies \( J^2_r \subset I^2_r \) and 1.7. gives the result. Let \( q : L_r(\mu) \to \mathbb{K} \) be positive and \( U \in \mathcal{L}(L_r(\mu); L_{r'}(\mu)) \) a positive operator with \( q(f) = \langle Uf, f \rangle \).

Maurey's factorization theorem (see e.g. [DF, 18.9. and 18.10.]) gives that \( U \) factors as follows (with linear, continuous \( R, S \) and a finite measure \( \mu_0 ) \)

\[
\begin{array}{c c c c c c}
 & & L_r(\mu) & U & L_{r'}(\mu) & \\
& R & ./. & & S & \\
L_r(\mu_0) & \hookleftarrow & L_{r'}(\mu_0) & \\
\end{array}
\]

hence \( q(f) = \langle Uf, f \rangle = \int Rf(w) \cdot S'f(w) \mu_0(dw) \). Using \( [(R + S')(f)]^2 = [Rf]^2 + [S'f]^2 + 2Rf \cdot S'f \) one obtains

\[
q = \frac{1}{2} \left[ q^2_{\mu_0,r} \circ (R + S') - q^2_{\mu_0,r} \circ R - q^2_{\mu_0,r} \circ S' \right].
\]

All three terms on the right side are in \( I^2_r \), hence also \( q \).

In particular, \( I^2_r = (I^2_r)^{\text{max}} \) with equivalent norms.

**Problems.**

(a) Is \( (I^n_r, I^n_r) \) maximal?

(b) Does \( I^n_r = J^n_r \) hold for \( n > 2 \)?

(c) Is \( I^n_r = J^n_r \) for all \( n \geq 2 \)?
Note that the proof used that the maximal ideal of \textit{bilinear maps} which factor through a positive bilinear map \( \varphi : L_r(\mu) \times L_r(\mu) \to \mathbb{K} \) for some finite measure is associated (in the sense of 1.8) with the ideal \( \mathcal{J}_r^2 \) of \( 2 \)-homogeneous polynomials.

\textbf{4.5.} For \( r = \infty \) one has the

**Proposition.** \( T^n = T^n_\infty \subseteq L^n_\infty \) for all \( n \geq 2 \).

**Proof.** It is enough to show that for some finite and positive measure \( \mu \)
\[
L^n_\infty(\mu) = P^n(\mu) \subseteq T^n(\mu).
\]

Arguing by contradiction, one has for all such \( \mu \)
\[
\left( \otimes^{\alpha,s}_\varepsilon L_\infty(\mu) \right)' = T^n(\mu) = P^n(\mu) = \left( \otimes^{\alpha,s}_\pi L_\infty(\mu) \right)';
\]
these representations were already mentioned in 1.6. It follows that \( \varepsilon_s \) and \( \pi_s \) are equivalent on \( \otimes^{\alpha,s} L_\infty(\mu) \) for all finite \( \mu \). Since the full tensor product \( \otimes^{\alpha,s}_\beta L_\infty(\mu) \) is isomorphic to a complemented subspace of \( \otimes^{\alpha,s}_\alpha L_\infty(\mu) \) for \( \alpha, \beta \in \{(\pi_s, \pi_s), (\varepsilon_s, \varepsilon_s), (\pi_s, \varepsilon_s), (\varepsilon_s, \pi_s)\} \) (see e.g. [F1, 2.8. and 3.7.]) and \( L_\infty(\mu) \) is isomorphic to some \( L_\infty(\nu) \), one has that the injective and projective norms \( \varepsilon \) and \( \pi \) are equivalent on \( \otimes^n L_\infty(\mu) \).

For \( n = 2 \) this would imply that every operator \( L_\infty \to L_1 \) is integral – but this is false; one proof of this runs as follows: By maximality and \( L_\infty \)-local technique one had \( L_1 \circ L_\infty \subseteq \mathcal{I} \) (see 3.5. and 3.7. for the notation), hence the cyclic composition theorem [DF, 25.4.] gives \((^* \text{ for adjoint ideals})\)
\[
\mathcal{I}^* \circ L_1 \subseteq L_\infty^*.
\]

Since \( \mathcal{I}^* = \mathcal{L} \) and \( \mathcal{L}^*_\infty = P_1 \) (the ideal of 1-summing operators) one obtains \( L_1 \subseteq P_1 \), but \( id_{L_1[0,1]} \) is not power-compact.

For \( n \geq 3 \) a result of John [J] says that \( \otimes^n E = \otimes^n_\varepsilon E \) holds topologically only if \( \dim E < \infty \).

\( \square \)

**Problem.** Are the inclusions \( \mathcal{I}^n_r \subseteq L^n_r \) also strict for \( r < \infty \)?

\textbf{4.6.} Let \( \mu \) be a finite, positive measure, \( s = \left( \frac{L}{n} \right)' \) and \( g \in L_s(\mu) \), then
\[
q^s_g : L_r(\mu) \to \mathbb{K} \quad q^s_g(f) := \int f^n g \, d\mu
\]
is well-defined and \( \|q^s_g\| = \|g\|_{L_s(\mu)} \) (recall that \( (\mathbb{K}, n) \) is “good”). Standard techniques (as in the proof [DF, 18.10]) show that \( I^s_r(q^s_g) \leq \|q^s_g\| = \|g\|_{L_s(\mu)} \) which implies the
Remark. $q \in \mathcal{P}^n(E)$ is in $\mathcal{I}^n_r(E)$ if and only if it has a factorization

$$q : E \xrightarrow{T} L_\ast(\mu) \xrightarrow{q^\ast_g} \mathbb{K}$$  \hspace{1cm} (\ast)$$

for some positive and finite measure $\mu$ and $q \in L_\ast(\mu)$. In this case:

$$\mathcal{I}^n_r(q) = \inf \left\{ \|T\|^n \|g\|_{L_\ast(\mu)} \mid q = q^\ast_g \circ T \text{ as in } (\ast) \right\}$$

4.7. It was already mentioned that for every maximal normed ideal $Q$ of $n$-homogeneous polynomials there is a unique finitely generated $s$-tensor norm $\alpha$ with

$$Q(E) = \left( \bigotimes^n_{\alpha} E \right)$$

Problem. Describe the $s$-tensor norms which are associated with $\mathcal{L}^n_r$ and $\mathcal{J}^n_r$.

It is rather immediate to see that the injective hull of the projective $s$-norm $\pi_s$ is associated with $\mathcal{L}_{\infty}^n$, the extendible polynomials.

References


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