The extension theorem for norms on symmetric tensor products of normed spaces

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To Professor Manuel Valdivia on the occasion of his seventieth birthday.

Abstract

It is shown that every $s$-tensor norm on $n$-th symmetric tensor products of normed spaces ($n$ fixed) is equivalent to the restriction on symmetric tensor products of a tensor norm (in the sense of Grothendieck) on "full" $n$-fold tensor products of normed spaces. As a consequence a large part of the isomorphic theory of norms on symmetric tensor products can be deduced from the theory of "full" tensor norms, which usually is easier to handle. Dually, the isomorphic theory of maximal normed ideals of $n$-homogeneous polynomials can be treated, to a certain extent, through the theory of maximal normed ideals of $n$-linear functions or mappings.

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1. Introduction and definitions

1.1. Symmetric and full tensor products of vector spaces (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are defined by universal properties (linearizing all symmetric $n$-linear or all $n$-linear mappings respectively) where $n \in \mathbb{N}$ is fixed (the case $n = 1$ is trivial). The $n$-th symmetric tensor product $\otimes^{n,s}E$ of a vector space $E$ can be obtained as the range $\text{im} \sigma^n_E$ of the symmetrization map $\sigma^n_E : \otimes^nE \rightarrow \otimes^nE$ (full $n$-fold tensor product) which linearizes

$$E^n \ni (x_1, \ldots, x_n) \mapsto x_1 \vee \cdots \vee x_n := \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)} =$$

$$= \frac{1}{n!} \int_{\Omega} \varepsilon_1(w) \cdots \varepsilon_n(w) \otimes^n \left[ \sum_{k=1}^n \varepsilon_k(w)x_k \right] P(dw) \in \otimes^nE$$

where $S_n$ denotes the group of permutations of $\{1, \ldots, n\}$ and $(\Omega, P)$ is a probability space with $\varepsilon_k : \Omega \rightarrow \mathbb{K}$ being stochastically independent, normalized ($\int_{\Omega} |\varepsilon_k|^2 dP = 1$) and centralized ($\int_{\Omega} \varepsilon_k dP = 0$) variables. The injection

$$\otimes^{n,s}E := \text{im} \sigma^n_E = \text{span} \{ \otimes^n x \mid x \in E \} \hookrightarrow \otimes^nE$$

will be denoted by $i^n_E$ and $\sigma^n_E : \otimes^nE \rightarrow \otimes^{n,s}E$ is the canonical projection (from now on $\sigma^n_E$ is considered as a map onto $\otimes^{n,s}E$); clearly $\sigma^n_E \circ i^n_E = \text{id}_{\otimes^{n,s}E}$. If $(E, F)$ is a separating dual system, then the following diagrams of natural maps...
Approximation property (proof as in [3, 13.2] for \(\beta\) codimensional subspaces of \(E\) — the latter using spaces without m.a.p. (see [3, 16.2] for all norms). Equivalently it is enough that these norms from the spaces is replaced by the class \(\text{FIN}\) of all finite-dimensional normed spaces. To distinguish given “full” tensor norm \(\beta\) be defined in a moment) it might be helpful to label them as “full tensor norms”. For a \((E)\) insted of \(\varnothing\) (where \(\text{FIN}(E)\) were introduced by R. Ryan [10] to Functional Analysis. [5] for details on the algebraic theory of symmetric tensor products; they commute. See [5] for details on the algebraic theory of symmetric tensor products; they were introduced by R. Ryan [10] to Functional Analysis.

1.2. A tensor norm \(\beta\) of order \(n\) assigns to each \(n\)-tuple \((E_1, \ldots, E_n)\) of normed spaces a norm \(\beta(\cdot; E_1, \ldots, E_n)\) on \(\otimes(E_1, \ldots, E_n)\) (notation \(\otimes_\beta(E_1, \ldots, E_n)\) or \(\otimes_{\beta, j=1}^n E_j\) or \(\otimes_\beta^n E\) if all \(E_j = E\) such that

1. \(\varepsilon \leq \beta \leq \pi\) where \(\varepsilon\) and \(\pi\) are the natural injective and projective norms.
2. The metric mapping property: If \(T_j \in \mathcal{L}(E_j; F_j)\), then

\[
\| \otimes_{j=1}^n T_j : \otimes_{\beta, j=1}^n E_j \to \otimes_{\beta, j=1}^n F_j \| = \| T_1 \| \cdots \| T_n \| .
\]

Equivalently it is enough that

1. \(\beta(\cdot; E_1, \ldots, E_n)\) is a seminorm for all normed spaces \(E_1, \ldots, E_n\).
2. \(\beta(\otimes^n 1; \mathbb{K}, \ldots, \mathbb{K}) = 1\)
3. \(\beta(\otimes^n; \mathbb{K}, \ldots, \mathbb{K}) = 1\)
4. \(\beta(\otimes^n; E_1, \ldots, E_n)\) is a seminorm for all normed spaces \(E_1, \ldots, E_n\).

It is clear that the same definitions can be made if the class \(\text{NORM}\) of all normed spaces is replaced by the class \(\text{FIN}\) of all finite dimensional normed spaces. To distinguish these norms from the \(s\)-tensor norms (on symmetric tensor products tensor products to be defined in a moment) it might be helpful to label them as “full tensor norms”. For a given “full” tensor norm \(\beta\) of order \(n\) one can define the tensor norms

\[
\bar{\beta}(z; E_1, \ldots, E_n) := \inf \{ \beta(z; M_1, \ldots, M_n) \mid M_j \in \text{FIN}(E_j), z \in \otimes_{j=1}^n M_j \}
\]

\[
\bar{\beta}(\cdot; E_1, \ldots, E_n) := \sup \{ \beta((\otimes_{j=1}^n Q_{L_{j}}^E)(z); E_1/L_1, \ldots, E_n/L_n) \mid L_j \in \text{COFIN}(E_j) \}
\]

(where \(\text{FIN}(E)\) is the set of finite dimensional subspaces of \(E\), \(\text{COFIN}(E)\) the set of finite codimensional subspaces of \(E\) and \(Q_{L_j}^E : E \to E/F\) the natural quotient mapping); note that for these constructions it is enough to know \(\beta\) on finite dimensional spaces. The mapping property implies \(\bar{\beta} \leq \beta \leq \bar{\beta}\) and all three coincide if \(E_1, \ldots, E_n\) have the metric approximation property (proof as in [3, 13.2] for \(n = 2\)). \(\beta\) is called finitely generated if \(\bar{\beta} = \bar{\beta}\) and cofinitely generated if \(\bar{\beta} = \beta\). It is easy to see that \(\varepsilon = \bar{\varepsilon} = \bar{\varepsilon}\) and \(\pi = \bar{\pi} = \bar{\pi}\) — the latter using spaces without m.a.p. (see [3, 16.2] for \(n = 2\)). If \(\beta\) is a tensor norm of order \(n\), then the dual tensor norm \(\beta'\) is defined on \(\text{FIN}\) by

\[
\otimes_{\beta'}(M_1, \ldots, M_n) := \left[ \otimes_\beta(M_1', \ldots, M_n') \right]'
\]
and on NORM by the finite hull $\overline{\beta}$. A tensor norm $\beta$ is called injective (resp. semi-injective) if
\[
\otimes_\beta(F_1, \ldots, F_n) \to \otimes_\beta(E_1, \ldots, E_n)
\]
is a metric (resp. isomorphic) injection whenever $F_j \subset E_j$ are subspaces. In the case of semi-injectivity it is easy to see that there are universal constants for the equivalence of the norms on $\otimes(F_1, \ldots, F_n)$. The tensor norm $\beta$ is called projective (resp. semi-projective) if the natural map
\[
\otimes^n_{j=1} Q^E_{F_j} : \otimes^n_\beta(E_1, \ldots, E_n) \to \otimes^n_{\beta}(E_1/F_1, \ldots, E_n/F_n)
\]
is a metric surjection (resp. open) whenever $F_j \subset E_j$ are closed subspaces; again there are universal constants in the semi-projective case.

1.3. The theory of “full” tensor norms is well-developed in the case $n = 2$ (see e.g. [3]), it is due to Grothendieck [8] and, up to a certain extent, also to Schatten [11]. Many results are easily extended from 2 to $n$.

1.4. Again being $n \in \mathbb{N}$ fixed, an $s$-tensor norm $\alpha$ of order $n$ assigns to each normed space $E$ a norm $\alpha(\cdot; \otimes^n_{s}E)$ on $\otimes^{n,s}E$ such that

1. $\varepsilon_s \leq \alpha \leq \pi_s$

2. The metric mapping property: If $T \in \mathcal{L}(E; F)$, then

\[
\| \otimes^n_{s} T : \otimes^n_{\alpha}E \to \otimes^n_{s}F \| = \| T \|^n.
\]

The theory of the natural projective $s$-tensor norm $\pi_s$ and natural injective $s$-tensor norm $\varepsilon_s$ is presented e.g. in [5]. As in the case of “full” tensor norms there is a useful test: $\alpha$ is an $s$-tensor norm of order $n$ if

1. $\alpha(\cdot; \otimes^n{s}E)$ is a seminorm on $\otimes^{n,s}E$ (for all normed spaces $E$)

2. $\alpha(\otimes^n{1}; \otimes^n{s}K) = 1$

3. Like (2) with $\leq \| T \|^n$.

Restricting $E$ to be in FIN (or on Hilbert spaces) one obtains the definition of an $s$-tensor norm of order $n$ on FIN (or on the class of all Hilbert spaces). Having the definitions for full tensor norms in mind it is clear how to define the finite hull $\alpha$, the cofinite hull $\alpha'$ and the dual norm $\alpha'$; note that for $M \in \text{FIN}$

\[
(\otimes^n_{s} M)' = \otimes^{n,s} M'
\]

(by definition) and hence

\[
|\langle z, z' \rangle| \leq \alpha(z; \otimes^n{s}M)\alpha'(z'; \otimes^n{s}M')
\]

for all $z \otimes^{n,s} M$ and $z' \in \otimes^n{s}M'$; this sort of trace-duality is often useful. It follows from the definition of $\varepsilon_s$ (see e.g. [5, 3.1.]) that $\pi'_s = \varepsilon_s$. Moreover, it is obvious how to define $\alpha$ to be finitely generated, cofinitely generated, injective, semi-injective, projective and semi-projective.

An introduction to the theory of $s$-tensor norms will be presented in [6]; in this paper I shall need only the basic definitions.
2. The norm extension theorem

2.1. Fix \( n \in \mathbb{N} \). If \( \beta \) is a “full” tensor norm of order \( n \), then

\[
\beta_s(z; \otimes^n sE) := \beta(t^n_E(z); nE)
\]

(interprete \( nE \) as \( E, \ldots, E \) (\( n \)-times)) defines an \( s \)-tensor norm of order \( n \) with \( \varepsilon_s \leq \beta_s \leq \pi_s \). Since \( \pi_s \leq \pi_s \) but \( \pi_s \neq \pi_s \) (and \( \varepsilon_s \neq \varepsilon_s \); see [5]) not all \( s \)-tensor norms are of this form. However, for every \( s \)-tensor norm \( \alpha \) there is a full \( \beta \) such that \( \alpha \) and \( \beta \) are equivalent (notation: \( \alpha \sim \beta_s \)): This is the main content of the norm extension theorem which will be proved. \( \beta \) can be chosen to be symmetric, i.e. the natural map \( R_\eta : \otimes_\beta(E_1, \ldots, E_n) \rightarrow \otimes_\beta(E_{\eta(1)}, \ldots, E_{\eta(n)}) \) is an isometry (onto) for all \( \eta \in S_n \). Clearly the projective norm \( \pi \) and the injective norm \( \varepsilon \) are symmetric. For \( n = 2 \) the norms \( w_2 \) and \( w_2' \) are symmetric.

2.2. It is worthwhile to have good information about the constants in the norm extension theorem. For this define for an \( s \)-tensor norm \( \alpha \)

\[
K_2(\alpha) := \sqrt{n!} \alpha(e_1 \vee \cdots \vee e_n; \otimes^n s\ell^n_2)
\]

where the \( e_j \) are the unit vectors in \( \ell^n_2 \).

Lemma. \( \left( \frac{n!}{n} \right)^{1/2} = K_2(\varepsilon_s) \leq K_2(\alpha) \leq K_2(\pi_s) = \left( \frac{n^n}{n!} \right)^{1/2} \).

**Proof:** It is clear that \( K_2(\varepsilon_s) \leq K_2(\alpha) \leq K_2(\pi_s) \) for all \( s \)-tensor norms \( \alpha \). For an upper estimate of \( K_2(\pi_s) \) use the Rademacher functions \( r_k : [0,1] \rightarrow \{-1,1\} \) in the polarization formula

\[
e_1 \vee \cdots \vee e_n = \frac{1}{n!} \int_0^1 r_1(t) \cdots r_n(t) \otimes^n (r_1(t), \ldots, r_n(t))dt
\]

hence

\[
\pi_s(e_1 \vee \cdots \vee e_n; \otimes^n s\ell^n_2) \leq \frac{1}{n!} \int_0^1 \|(r_1(t), \ldots, r_n(t))\|_{\ell^n_2} dt \leq \frac{(\sqrt{n})^n}{n!}.
\]

For \( \varepsilon_s \) one obtains:

\[
\varepsilon_s(e_1 \vee \cdots \vee e_n; \otimes^n s\ell^n_2) = \sup\{|\langle e_1 \vee \cdots \vee e_n; \otimes^n x' \rangle | x' \in B_{\ell^n_2} \} = \sup\{ |x'_1 \cdots x'_n| | \sum_{k=1}^n |x'_k|^2 = 1 \} = n^{-n/2}
\]

– where the last equality can be proved using Lagrange multipliers. Now observe

\[
\langle e_1 \vee \cdots \vee e_n, e_1 \vee \cdots \vee e_n \rangle = \left( \frac{1}{n!} \right)^2 \sum_{\eta, \sigma \in S_n} \langle e_{\eta(1)} \otimes \cdots \otimes e_{\eta(n)}, e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)} \rangle
\]

\[
= \frac{1}{n!}
\]
hence the “trace-duality” \((\otimes^{n,s}_{\pi_n} \Pi_{2}^n)') = \otimes^{n,s}_{\pi_n} \Pi_{2}^n\) implies
\[
\frac{1}{n!} = (e_1 \lor \cdots \lor e_n, e_1 \lor \cdots \lor e_n) \leq \pi_s(e_1 \lor \cdots \lor e_n; \otimes^{n,s}_{\Pi_{2}^n} e_s(e_1 \lor \cdots \lor e_n; \otimes^{n,s}_{\Pi_{2}^n})
\leq \frac{n^{n/2}}{n!} n^{n/2} = \frac{1}{n!}
\]
and therefore \(\pi_s(e_1 \lor \cdots \lor e_n; \otimes^{n,s}_{\Pi_{2}^n}) = \frac{n^{n/2}}{n!}\).

The trace-duality gives \(K_2(\alpha)K_2(\alpha') \geq 1\) for all \(s\)-tensor norms \(\alpha\) and it would be interesting to check whether equality holds (as in the case of \(\alpha = \varepsilon_s\)).

2.3. Everything is prepared to state and prove the

**Norm Extension Theorem.** For every \(s\)-tensor norm \(\alpha\) of order \(n\) there is a full symmetric tensor norm \(\beta\) of order \(n\) with \(\beta|_s\) being equivalent to \(\alpha\). More precisely: there is a construction which gives for every \(s\)-tensor norm \(\alpha\) of order \(n\) a full symmetric tensor norm \(\Phi(\alpha)\) of order \(n\) such that

1. \(\|\sigma^n_{\Phi(\alpha)} : \otimes^n_{\Phi(\alpha)} E \rightarrow \otimes^n_{\alpha} E\| \leq \left(\frac{n^n}{n!}\right)^{1/2} K_2(\alpha) \leq \frac{n^n}{n!} \) for all normed spaces \(E\).
2. \(\|\iota^n_{\Phi(\alpha)} : \otimes^n_{\alpha} E \rightarrow \otimes^n_{\Phi(\alpha)} E\| \leq \left(\frac{n^n}{n!}\right)^{1/2} K_2(\alpha)^{-1} \leq \frac{n^n}{n!} \) for all normed spaces \(E\).
3. In particular:
\[
\frac{n!}{n^n} \Phi(\alpha)|_s \leq \left(\frac{n^n}{n!}\right)^{1/2} K_2(\alpha) \Phi(\alpha)|_s \leq \alpha \leq \left(\frac{n^n}{n!}\right)^{1/2} K_2(\alpha) \Phi(\alpha)|_s \leq \frac{n^n}{n!} \Phi(\alpha)|_s
\]
4. If \(\alpha_1 \leq c \alpha_2\), then \(K_2(\alpha_1) \Phi(\alpha_1) \leq c K_2(\alpha_2) \Phi(\alpha_2)\).
5. If \(\alpha\) is finitely generated (resp. cofinitely generated), then \(\Phi(\alpha)\) is finitely generated (resp. cofinitely generated).
6. If \(\alpha\) is injective, then \(\Phi(\alpha)\) is injective.
7. If \(\alpha\) is semi-projective (resp. semi-injective), then \(\Phi(\alpha)\) is semi-projective (resp. semi-injective).
8. For the dual norm \(\alpha'\) one has \(\Phi(\alpha') \sim \Phi(\alpha)'\), more precisely
\[
\Phi(\alpha') \leq K_2(\alpha) K_2(\alpha') \Phi(\alpha') \leq n^{n/2} \Phi(\alpha)'\.
\]
9. If \(\gamma\) is a full symmetric tensor norm of order \(n\), then \(\Phi(\gamma)|_s \sim \gamma\); with constants:
\[
\frac{1}{\sqrt{n!}} K_2(\gamma)|_s \Phi(\gamma)|_s \leq \gamma \leq \sqrt{n!} K_2(\gamma)|_s \Phi(\gamma)|_s\).
PROOF: (a) If $E_1, \ldots, E_n$ are normed spaces, $\ell^2_2(E_j) := \ell^2(E_1, \ldots, E_n)$ and $P_k : \ell^2_2(E_j) \to E_k$ and $I_k : E_k \hookrightarrow \ell^2_2(E_j)$ the natural projections and injections, then it is straightforward to see that $\text{id}_{\otimes^n_{j=1} E_j} = Q_{E_1, \ldots, E_n} \circ J_{E_1, \ldots, E_n}$ where

$$J_{E_1, \ldots, E_n} : \otimes^1_{j=1} E_j \xrightarrow{\sqrt{n!} I_1 \otimes \cdots \otimes I_n} \otimes^n \ell^2_2(E_j) \xrightarrow{\sigma^{n}_J(E_j)} \otimes^n \ell^2_2(E_j)$$

$$Q_{E_1, \ldots, E_n} : \otimes^n \ell^2_2(E_j) \xrightarrow{\ell^n_2(E_j)} \otimes^n \ell^2_2(E_j) \xrightarrow{\sqrt{n!} P_1 \otimes \cdots \otimes P_n} \otimes^n_{j=1} E_j$$

(see [5, 1.10] for the origin of this factorization). Note that

$$J_{E_1, \ldots, E_n}(x_1 \otimes \cdots \otimes x_n) = \sqrt{n!} (x_1, 0, \ldots, 0) \vee \cdots \vee (0, \ldots, 0, x_n).$$

(b) The definition

$$\beta_\alpha (z; E_1, \ldots, E_n) := \alpha (J_{E_1, \ldots, E_n}(z); \otimes^n \ell^2_2(E_j))$$

gives a norm on $\otimes^n_{j=1} E_j$ which satisfies the metric mapping property (2') from 1.2.: to see this take $\|T_j : E_j \to F_j\| \leq 1$ and define $T : \ell^2_2(E_j) \to \ell^2_2(F_j)$ by $T(x_1, \ldots, x_n) := (T_1 x_1, \ldots, T_n x_n);$ then $\|T\| \leq 1$ and

$$\otimes^n_{j=1} E_j \xrightarrow{J_{E_1, \ldots, E_n}} \otimes^n \ell^2_2(E_j) \xrightarrow{\otimes^n T} \otimes^n \ell^2_2(F_j)$$

commutes. This shows $\|T_1 \otimes \cdots \otimes T_n : \cdots\| \leq 1$. To see that $\beta_\alpha$ is symmetric, note first that for every $\eta \in S_n$ the natural map $S_\eta : \ell^2_2(E_1, \ldots, E_n) \to \ell^2_2(E_{\eta(1)}, \ldots, E_{\eta(n)})$ is an isometry (onto); moreover, the diagram

$$\otimes^n_{j=1} E_j \xrightarrow{J_{E_1, \ldots, E_n}} \otimes^n \ell^2_2(E_j) \xrightarrow{\otimes^n S_n} \otimes^n \ell^2_2(E_{\eta(j)})$$

commutes, since

$$(\sqrt{n!})^{-1}(\otimes^n \eta S_\eta) J_{E_1, \ldots, E_n}(x_1 \otimes \cdots \otimes x_n) = (\otimes^n \eta S_\eta)(x_1, 0, \ldots, 0) \vee \cdots =
= S_\eta(x_1, 0, \ldots, 0) \vee \cdots \vee S_\eta(0, \ldots, 0, x_n) =
= (0, \ldots, x_1, \ldots, 0) \vee \cdots \vee (0, \ldots, x_n, \ldots, 0) =
= (x_{\eta(1)} ,0, \ldots, 0) \vee \cdots \vee (0, \ldots, 0, x_{\eta(n)}) =
= (\sqrt{n!})^{-1} J_{E_{\eta(1)} ,\ldots, E_{\eta(n)}}(x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)}).$$
The metric mapping property of $\alpha$ implies that $\beta_\alpha$ is symmetric. Since $\beta_\alpha(\otimes^n 1; \mathbb{K}, \ldots, \mathbb{K}) = K_2(\alpha)$ one obtains that

$$\Phi(\alpha) := K_2(\alpha)^{-1} \beta_\alpha$$

defines a full symmetric tensor norm of order $n$.

(c) To estimate $\sigma^n_E$ consider $S : \ell^n_2(E) \rightarrow E$ defined by $S(x_1, \ldots, x_n) := \sum_{k=1}^n x_k$; clearly $\|S\| = \sqrt{n}$. For $x_1, \ldots, x_n \in E$ one obtains

$$\sigma^n_E(x_1 \otimes \cdots \otimes x_n) = x_1 \vee \cdots \vee x_n = S(x_1, 0, \ldots, 0) \vee \cdots \vee S(0, \ldots, 0, x_n) = (n!)^{-1/2} [\otimes^n S] \circ J_{nE}(x_1 \otimes \cdots \otimes x_n) \in \otimes^n E$$

hence $\sigma^n_E = (n!)^{-1/2} [\otimes^n S] \circ J_{nE}$ which implies

$$\|\sigma^n_E : \otimes^n_\beta E \xrightarrow{J_{nE}} \otimes^n_\alpha \ell^n_2(E) \xrightarrow{(n!)^{-1/2} \otimes^n S} \otimes^n_\alpha E\| \leq (n!)^{-1/2} \|S\|^n = \left( \frac{n^n}{n!} \right)^{1/2}.$$

(d) To see the estimate for $\iota^n_E$ consider Rademacher functions $r_k : [0, 1] \rightarrow \{-1, +1\}$ and, for every $t \in [0, 1]$, the operators

$$D_t : \ell^n_2(E) \rightarrow \ell^n_2(E), \quad (x_1, \ldots, x_n) \mapsto (r_1(t)x_1, \ldots, r_n(t)x_n).$$

$$\Delta : E \rightarrow \ell^n_2(E), \quad x \mapsto (x, \ldots, x).$$

Thus $\|D_t\| = 1$ and $\|\Delta\| = \sqrt{n}$. For $x \in E$ one gets

$$J_{nE}(\otimes^n x) = \sqrt{n!} (x, 0, \ldots, 0) \vee \cdots \vee (0, \ldots, 0, x) =$$

$$= \frac{1}{\sqrt{n!}} \int_0^1 r_1(t) \cdots r_n(t) \otimes (r_1(t)x, \ldots, r_n(t)x) dt =$$

$$= \frac{1}{\sqrt{n!}} \int_0^1 r_1(t) \cdots r_n(t) [\otimes^n (D_t \circ \Delta)] \otimes^n x dt$$

and therefore for all $z \in \otimes^n E$

$$J_{nE}(z) = \frac{1}{\sqrt{n!}} \int_0^1 r_1(t) \cdots r_n(t) [\otimes^n (D_t \circ \Delta)] z dt \in \otimes^n \ell^n_2(E).$$

It follows for all $z \in \otimes^n E$ that

$$\beta_\alpha(\iota^n_E(z); nE) = \alpha(J_{nE}(z); \otimes^n \ell^n_2(E)) \leq \frac{1}{\sqrt{n!}} \|\Delta\|^n \alpha(z; \otimes^n E) \leq$$

$$\leq \left( \frac{n^n}{n!} \right)^{1/2} \alpha(z; \otimes^n E)$$

hence (2). Note that this gives in particular

$$\|Q_{E_1, \ldots, E_n} : \otimes^n \ell^n_2(E_j) \rightarrow \otimes^n_{\beta_{\alpha,j=1}} E_j\| \leq \left( \frac{n^n}{n!} \right)^{1/2} \cdot \sqrt{n!} = n^{n/2}$$

and $Q_{E_1, \ldots, E_n}$ is continuous; the fact that $\text{id} \otimes^n E_i = Q_{E_1, \ldots, E_n} \circ J_{E_1, \ldots, E_n}$ implies that $Q_{E_1, \ldots, E_n}$ is even open and hence $\Phi(\alpha)$ is equivalent to the quotient norm of $Q_{E_1, \ldots, E_n}$. 
(e) If $\alpha_1 \leq c \alpha_2$, then $\beta_{\alpha_1} \leq c \beta_{\alpha_2}$ which is (4).

(f) Assume that $\alpha$ is a finitely generated $s$-tensor norm. Since every $M \in \text{FIN} (\ell_2^n (E_j))$ is contained in some $\ell_2^n (M_j)$ with $M_j \in \text{FIN} (E_j)$ one obtains

$$\beta_\alpha (z; E_1, \ldots, E_n) = \alpha (J_{E_1, \ldots, E_n} (z); \otimes \alpha^n \ell_2^n (E_j)) =$$

$$= \inf \{ \alpha (J_{M_1, \ldots, M_n} (z); \otimes \alpha^n \ell_2^n (M_j) \mid M_j \in \text{FIN} (E_j), J \ldots (z) \in \otimes \alpha^n \ell_2^n (M_j) \} =$$

$$= \inf \{ \alpha (J_{M_1, \ldots, M_n} (z); \otimes \alpha^n \ell_2^n (M_j) \mid M_j \in \text{FIN} (E_j), z \in \otimes \alpha^n M_j \} =$$

$$= \inf \{ \beta_\alpha (z; M_1, \ldots, M_n) \mid M_j \in \text{FIN} (E_j), z \in \otimes \alpha^n M_j \}$$

which shows that $\Phi (\alpha)$ is finitely generated.

(g) If $\alpha$ is cofinitely generated take $L \in \text{COFIN} (\ell_2^n (E_j))$, recall $\ell_2^n (E_j)' = \ell_2^n (E_j')$ and choose $L_j \in \text{COFIN} (E_j)$ with

$$L \supset \ker [\ell_2^n (E_j) \rightarrow \ell_2^n (E_j'/L_j)] =: L_0 \in \text{COFIN} (\ell_2^n (E_j)).$$

The diagram

$$\begin{array}{ccc}
\otimes^n_{j=1} E_j & \xrightarrow{J_{E_1, \ldots, E_n}} & \otimes^n_{\alpha} \ell_2^n (E_j) \\
\otimes^n_{j=1} Q_{E_j} & \downarrow & \otimes^n_{\alpha} Q_{E_j}' \\
\otimes^n_{j=1} E_j/L_j & \xrightarrow{J_{E_1/L_1, \ldots, E_n/L_n}} & \otimes^n_{\alpha} \ell_2^n (E_j)/L_0 \xrightarrow{1} \otimes^n_{\alpha} \ell_2^n (E_j/L_j)
\end{array}$$

commutes which easily gives that $\beta_\alpha$ and hence $\Phi (\alpha)$ is cofinitely generated.

(h) If $\alpha$ is (semi-)injective it is immediate, by the construction, that $\beta_\alpha$ is (semi-)injective and hence $\Phi (\alpha)$. The fact that $\Phi (\alpha)$ is equivalent to the quotient norm of $Q_{E_1, \ldots, E_n}$ (see part (d) of this proof) implies easily that $\Phi (\alpha)$ is semi-projective if $\alpha$ is semi-projective.

(i) To show the statement (8) about the dual norms, take an $s$-tensor norm $\alpha$ and observe first, that $\Phi (\alpha')$ is finitely generated by (5); since $\Phi (\alpha')$ anyhow is finitely generated it is enough to show that $\Phi (\alpha')$ and $\Phi (\alpha')$ are equivalent on $\text{FIN}$ (with constants independent of the space): for this take $M := (M_1, \ldots, M_n) \in \text{FIN}^n$ and define, for convenience, $M' := (M_1', \ldots, M_n')$. Observe that

$$[I_k : M_k \rightarrow \ell_2^n (M)]' = [P_k : \ell_2^n (M') \rightarrow M_k']$$

and note that $\beta_{\alpha'} = K_2 (\alpha)^{-1} \Phi (\alpha)$. Dualizing

$$\otimes^n_{j=1} M' \xrightarrow{J_{M'}} \otimes^n_{\alpha} \ell_2^n (M') \xrightarrow{Q_{M'}} \otimes^n_{\alpha} M'.$$
to
\[ \otimes_{\beta_n}^n M \xrightarrow{Q_M'=J_M} \otimes_{\alpha}^n \ell_2^n(M) \xrightarrow{J_M'=Q_M} \otimes_{\beta_n}^n M \]
gives (see end of (d))
\[ \beta_{\alpha'}(z; M) = \alpha'(J_M(z); \otimes^n \ell_2^n(M)) \leq \|Q_M'\| \|\beta_{\alpha'}(z; M) \leq n^{n/2} \beta_{\alpha'}(z; M) \]
\[ \beta_{\alpha'}(z; M) = \beta_{\alpha'}(J_M'; J_M(z); M) \leq \|J_M'\| \|\alpha'(J_M(z); \otimes^n \ell_2^n(M)) = \beta_{\alpha'}(z; M) \]
which implies (8).

(j) Finally let \( \gamma \) be a full symmetric tensor norm, then \( c_E := \|\sigma_E^n : \otimes^n E \rightarrow \otimes^n \ell_2^n E\| \leq 1 \) by the symmetry. Define
\[ \gamma_1 := \beta_{\gamma_{|_s}} = K_2(\gamma_{|_s}) \Phi(\gamma_{|_s}), \]
then
\[ \gamma_1(z; E_1, \ldots, E_n) \leq c_{\ell_2^n(E)} \sqrt{n!} \gamma([I_1 \otimes \cdots \otimes I_n](z); \ell_2^n(E_1), \ldots, \ell_2^n(E_n)) \leq \sqrt{n!} \gamma(z; E_1, \ldots, E_n) \]
hence \( \gamma_1 \leq \sqrt{n!} \gamma \) and \( \Phi(\gamma_{|_s}) \leq K_2(\gamma_{|_s})^{-1} \sqrt{n!} \gamma \). On the other hand
\[ \gamma(z; E_1, \ldots, E_n) = \gamma(Q_{E_1,\ldots,E_n} \circ J_{E_1,\ldots,E_n}(z); E_1, \ldots, E_n) \leq \sqrt{n!} \gamma_{|_s}(J_{E_1,\ldots,E_n}(z); \otimes^n \ell_2^n E_j) = \sqrt{n!} \gamma_1(z; E_1, \ldots, E_n) \]
hence \( \gamma \leq \sqrt{n!} K_2(\gamma_{|_s}) \Phi(\gamma_{|_s}) \).

2.4. Some comments on the construction: it is clear that one may take \( \ell_p^n(E) \) in the construction (or any symmetric norm on \( \mathbb{R}^n \) instead of the \( \ell_2 \)-norm); I took \( p = 2 \) to facilitate the dualization.

2.5. Taking \( \tilde{\beta}_n \) to be the quotient norm of \( Q_{E_1,\ldots,E_n} \) would give (after normalization as in part (b) of the proof) a full symmetric tensor norm \( \Psi(\alpha) \) of order \( n \), equivalent to \( \Phi(\alpha) \) (see the end of (d) of the proof in 2.3.), satisfying (1)–(5), (7)–(9), (with other constants, a priori), and: If \( \alpha \) is projective, then \( \Psi(\alpha) \) is projective. Is \( \Psi(\alpha) \neq \Phi(\alpha) \)?

2.6. Note that (9) (and (4)) imply that two symmetric full tensor norms (of order \( n \)) are equivalent if they are equivalent on symmetric tensor products. In other words: There is (up to equivalence) at most one full symmetric tensor norm extending a given s-tensor norm.

2.7. Since
\[ \frac{n^n}{n!} = \|\sigma_E^n : \otimes^n \ell_1 \rightarrow \otimes^n \ell_1\| \leq \|\sigma_{\Phi(\alpha)}^n : \otimes^n \ell_1 \rightarrow \otimes^n \ell_1\| \leq \frac{n^n}{n!} \]
\[ \frac{n^n}{n!} = \|\ell_{\Phi(\alpha)}^n : \otimes^n c_0 \rightarrow \otimes^n c_0\| \leq \|\ell_{\Phi(\alpha)}^n : \otimes^n c_0 \rightarrow \otimes^n \ell_1\| \leq \frac{n^n}{n!} \]
(see e.g. [5, 2.1., 2.3. and 5.3]) it follows that the constant \( \frac{n^n}{n!} \) is best possible.
2.8. I do not know whether any $s$-tensor norm $\alpha$ with $\varepsilon|_{s_0} \leq \alpha \leq \pi|_{s_0}$ can be extended to a full norm $\gamma$ with $\gamma|_{s_0} = \alpha$.

2.9. For the investigation of individual spaces the constants

\[
c(n, \alpha, E) := \| \sigma^n_E : \otimes^n_{\Phi(\alpha)} E \rightarrow \otimes^n_{\alpha} E \| \leq \left( \frac{n^n}{n!} \right)^{1/2} K_2(\alpha) \leq \frac{n^n}{n!}
\]

\[
d(n, \alpha, E) := \| \nu^n_E : \otimes^n_{\alpha} E \rightarrow \otimes^n_{\Phi(\alpha)} E \| \leq \left( \frac{n^n}{n!} \right)^{1/2} K_2(\alpha)^{-1} \leq \frac{n^n}{n!}
\]

may be of interest.

3. Some applications

3.1. Every continuous $n$-homogeneous polynomial on a normed space $E$, notation: $q \in \mathcal{P}^n(E)$, has a canonical extension $q \in \mathcal{P}^n(E'')$, usually called the Aron-Berner extension (see e.g. [5, 6.5.]). Let us use the identification $\mathcal{P}^n(E) \overset{1}{\rightarrow} (\otimes^n_{\varepsilon} E)'$, $q \sim q^L$. It would be interesting to know whether

\[
\| q^L \|_{(\otimes^n_{\varepsilon} E)'} = \| q^L \|_{(\otimes^n_{\alpha} E)'} \in [0, \infty]
\]

holds. For $\alpha = \pi_s$ (Davie-Gamelin [4]) and $\alpha = \varepsilon_s$ (Carando-Zalduendo [2], see [5] for an alternative proof) this is true, but not at all trivial.

**Isomorphic Extension Lemma.** Let $\alpha$ be a finitely generated $s$-tensor norm of order $n$, $E$ normed and $q \in \mathcal{P}^n(E)$. Then $q^L \in (\otimes^n_{\alpha} E)'$ if and only if $q^L \in (\otimes^n_{\alpha} E'')'$. 

**Proof:** Since $E \hookrightarrow E''$, the metric mapping property implies one direction. For the other direction take a finitely generated full tensor norm $\beta$ of order $n$ such that $\beta|_{s_0} \sim \alpha$ (it exists due to the norm extension theorem). It can be seen (more or less as in the case $n = 2$, see [3, 13.2.]) that the canonical Arens-extension $\varphi$ (see [5, 6.1.]) is in $(\otimes^n_{\beta} E'')'$ if $\varphi$ is in $(\otimes^n_{\beta} E)'$. Setting $\varphi := q^L \circ \sigma^n_E$, gives $\varphi = \varphi \circ \nu^n_{E''}$ hence the result follows from the properties of $\beta$. 

3.2. It is worthwhile to mention that a normed ideal $\mathcal{Q}$ of $n$-homogeneous scalar-valued polynomials is maximal if and only if it is of the form

\[
\mathcal{Q}(E) \overset{1}{\rightarrow} (\otimes^n_{\alpha} E)'
\]

for some finitely generated $s$-tensor norm $\alpha$ of order $n$; see [7]. The maximal normed ideals $\mathcal{A}$ of $n$-linear continuous functionals are of the form

\[
\mathcal{A}(E_1, \ldots, E_n) \overset{1}{\rightarrow} (\otimes_{\beta} (E_1, \ldots, E_n))'
\]

(for all Banach spaces $E_j$) for some finitely generated full tensor norm of order $n$ (see also [7]). The norm extension theorem (use in particular that $\Phi(\alpha)$ is finitely generated if $\alpha$ is) easily gives the
Proposition. For every maximal normed ideal $Q$ of $n$-homogeneous scalar-valued polynomials there is a maximal normed ideal $A$ of $n$-linear functionals such that $q \in P^n(E)$ (with associated symmetric $n$-linear form $\tilde{q}$) is in $Q(E)$ if and only if $\tilde{q} \in A(E, \ldots, E)$.

Checking the constants gives (if $\alpha$ is the “associated” finitely generated $s$-tensor norm to $Q$ and $A$ the ideal “associated” with the full tensor norm $\Phi(\alpha)$)

$$\|\tilde{q}\|_A \leq c(n, \alpha, E) \|q\|_Q$$

$$\|q\|_Q \leq d(n, \alpha, E) \|\tilde{q}\|_A$$

3.3. If $\beta$ is a finitely generated full tensor norm of order 2, then the canonical map $\ast : \otimes^2_{\alpha}E \rightarrow \otimes^2_{\alpha}E$ is injective if $E$ has the approximation property ($\tilde{\ast}$ stands for the completion; see e.g. [3, 17.20.] for a proof). The norm extension theorem easily gives: If $\alpha$ is a finitely generated $s$-tensor norm of order 2 and $E$ a Banach space with the approximation property, then the canonical map $\ast : \otimes^2_{\alpha}E \rightarrow \otimes^2_{\alpha}E$ is injective. It is likely but not known whether $\ast$ (and hence $\ast$) is injective also for arbitrary $n$. If $E$ has even the metric approximation property it is an easy consequence of the duality theory of $s$-tensor norms (to be presented in [6]) that $\ast$ is injective for all $n$.

3.4. In the proof of $\Phi(\gamma|_{s}) \sim \gamma$ in the norm extension theorem it was not used that $\gamma$ is symmetric, only that $\|\sigma^n_{E} : \otimes^n_{\gamma}E \rightarrow \otimes^n_{\gamma|_{s}}E\| \leq 1$.

Proposition. If $\gamma$ is a full tensor norm of order $n$ such that $\sigma^n_{E} : \otimes^n_{\gamma}E \rightarrow \otimes^n_{\gamma|_{s}}E$ is continuous for all normed spaces $E$, then there is a full symmetric tensor norm $\beta$ of order $n$ equivalent to $\gamma$.

Proof: It is easy to see that there is a universal constant $c$ with $\|\sigma^n_{E} \ldots\| \leq c$. Now use just the part (j) of the proof of the norm extension theorem.

Note that $\beta$ can be chosen finitely generated, cofinitely generated, injective or projective if $\gamma$ is.

Corollary. For every full tensor norm $\gamma$ of order $n$ the following statements are equivalent:

1. For every normed space $E$ the symmetrization map $\sigma^n_{E} : \otimes^n_{\gamma}E \rightarrow \otimes^n_{\gamma|_{s}}E$ is continuous.

2. For every permutation $\eta \in S_n$ and all normed spaces $E_1, \ldots, E_n$ the natural map

$$R_{\eta} : \otimes_{\gamma}(E, \ldots, E_n) \rightarrow \otimes_{\gamma}(E_{\eta(1)}, \ldots, E_{\eta(n)})$$

is continuous.

Note that, for $n = 2$, Cohen’s tensor norm $w_1$ is not symmetric since $w_1$ is associated with the operator ideal of 1-factorable operators and $w^1_1 = w_\infty$ with the one of $\infty$-factorable operators (see [3, 17.8. and 17.12] for details). If follows that $\sigma^n_{E} : \otimes^n_{w_1}E \rightarrow \otimes^n_{w_1}E$ is, in general, not continuous.

3.5. A direct proof of the non-trivial part (1) of this latter result runs as follows:
For every locally convex space \( E \) the symmetrization map \( \sigma^n_E : \otimes^n E \to \otimes^n E \) is continuous.

(2) For every \( \eta \in S_n \) and locally convex spaces \( E_1, \ldots, E_n \) the natural map \\
\( \otimes_\eta(E_1, \ldots, E_n) \to \otimes_\eta(E_{\eta(1)}, \ldots, E_{\eta(n)}) \) is continuous (hence a homomorphism).

3.6. If \( \beta \) is a full tensor norm of order \( n \), one can construct a tensor topology of order \( n \) (the so-called tensor topology associated with \( \beta \)) which, for each \( n \)-tuple of locally convex spaces \( (E_1, \ldots, E_n) \), is defined by the seminorms \( \otimes_\beta(p_1, \ldots, p_n) \)

\[
\otimes_\beta(p_1, \ldots, p_n) := \beta((\otimes_{j=1}^n Q_{j})(z)); E_1/\ker p_1, \ldots, E_n/\ker p_n
\]

(\( p_j \) runs through a basis of continuous seminorms on \( E_j \) and \( Q_{p_j} : E_j \to E_j/\ker p_j \) are the natural quotient maps); these topologies were introduced by Harksen in 1979.
n = 2 (see [9] and [3, §35]). Notation: \( \otimes \beta (E_1, \ldots, E_n) \) or \( \otimes^n \beta E \) if \( E = E_1 = \cdots = E_n \). If \( \alpha \) is an \( s \)-tensor norm of order \( n \) the same idea

\[
(\otimes^{n,s}_\alpha p)(z) := \alpha((\otimes^{n,s}Q_p)(z); \otimes^{n,s}E/\ker p)
\]

defines a locally convex topology on \( \otimes^{n,s}_\alpha E \); notation: \( \otimes^{n,s}_\alpha E \).

**Proposition.** For each \( s \)-tensor norm \( \alpha \) of order \( n \) there is a full tensor norm \( \beta \) of order \( n \) such that for all locally convex spaces \( E \) the space \( \otimes^{n,s}_\alpha E \) is a complemented topological subspace of \( \otimes^n \beta E \) (via \( \iota^E_n \)).

This is an immediate consequence of the norm extension theorem. The properties (4)–(9) have consequences for \( \otimes^n \beta \): For example, \( \otimes^n \beta \) respects subspaces/quotients topologically if \( \otimes^{n,s}_\alpha \) does.

**REFERENCES**