Invariant Distances and Metrics in Complex Analysis — revisited

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In 1993, the authors published their book “Invariant Distances and Metrics in Complex Analysis”, in which they discussed the state of affairs in the domain covered by the title of that book. In the meantime, some open questions mentioned in the book have been solved, more explicit formulas for different invariant functions on certain concrete domains were found (see also [Kob 1998]). Moreover, the classical Green function became important in studying Bergman completeness which finally led to the result that any hyperconvex bounded domain is Bergman complete. Simultaneously, a new development started, namely the study of the Green functions with multipoles. This led to the creation of a lot of new objects. Recently, there was the surprising example of the symmetrized bidisc which initiated a lot of new activities. The symmetrized bidisc is not biholomorphically equivalent to a convex domain but, nevertheless, its Carathéodory distance and its Lempert function coincide. Hence, it becomes again an interesting question, for which domains these two objects are equal.

The main idea of this work is to describe what happened during the last 10 years in the area. The main source is our old book; in particular, if we quote a result before 1993 we send the reader to our book and not to the original source. This kind of quotation seems to be easier for the authors as well as for the reader. As always, the authors have to apologize for their choice of the material they present. Of course, it reflects their personal taste.

At the end of the writing process a lot of our colleagues helped us to find typographical and mathematical errors in the manuscript and to essentially improve our presentation. We like to thank them all, in particular, thanks are due to our colleagues Z. Blocki, A. Edigarian, P. Jucha, N. Nikolov, H. Youssfi, P. Zapalowski, and W. Zwonek. Nevertheless, the authors are responsible for all mistakes which remained.

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CHAPTER 1

Holomorphically invariant objects

1.1. Holomorphically contractible families of functions

Let us begin with the following definition of a holomorphically contractible family (cf. [J-P 1993], § 4.1).

Definition 1.1.1. A family \((d_G)_G\) of functions \(d_G : G \times G \rightarrow \mathbb{R}_{+}\) (1), where \(G\) runs over all domains \(G \subset \mathbb{C}^n\) (with arbitrary \(n \in \mathbb{N}\)), is said to be holomorphically contractible if the following two conditions are satisfied:

(A) for the unit disc \(E \subset \mathbb{C}\) we have
\[d_E(a,z) = m_E(a,z) := \left| \frac{z - a}{1 - \overline{a}z} \right|, a, z \in E\]

(the function \(m_E : E \times E \rightarrow [0,1)\) is called the Möbius distance),

(B) for any domains \(G \subset \mathbb{C}^n\), \(D \subset \mathbb{C}^m\), every holomorphic mapping \(F : G \rightarrow D\) is a contraction with respect to \(d_G\) and \(d_D\), i.e.
\[d_D(F(a),F(z)) \leq d_G(a,z), a, z \in G. \quad (1.1.1)\]

Notice that there is also another version of the definition of the holomorphically invariant family in which the normalization condition (A) is replaced by the condition

\[(A') d_E = p_E,\]

where
\[p_E := \frac{1}{2} \log \frac{1 + m_E}{1 - m_E}\]

is the Poincaré distance.

Both definitions are obviously equivalent in the sense that \((d_G)_G\) fulfills (A,B) iff the family \((\tanh^{-1} d_G)_G\) satisfies \((A',B)\). In our opinion the normalization condition (A) is more handy in calculations.

The following contractible families seem to be the most important.

- Möbius pseudodistance:
\[c^*_E(a,z) := \sup \{ m_E(f(a), f(z)) : f \in \mathcal{O}(G,E) \}\]
\[= \sup \{ |f(z)| : f \in \mathcal{O}(G,E), f(a) = 0 \}, \quad (a,z) \in G \times G;\]

the function \(c_G := \tanh^{-1} c^*_G\) is called the Carathéodory pseudodistance.

\[(1) A_+ := \{ x \in A : x \geq 0 \} (A \subset \mathbb{R}), A^n_+ := (A_+)^n, e.g. \mathbb{R}_+ = [0, +\infty), \mathbb{Z}_+ = \{0, 1, 2, \ldots\}, \mathbb{R}^n_+, \mathbb{Z}^n_+.\]
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• **Higher order Möbius function:**
  \[ m^{(k)}_G(a, z) := \sup \{|f(z)|^{1/k} : f \in \mathcal{O}(G, E), \, \text{ord}_a f \geq k\}, \quad (a, z) \in G \times G, \, k \in \mathbb{N}, \]
  where \( \text{ord}_a f \) denotes the order of zero of \( f \) at \( a \).

• **Pluricomplex Green function:**
  \[ g_G(a, z) := \sup \{u(z) : u : G \to [0, 1), \, \exists_{G=C(w,a)>0} \forall_{w \in G} : u(w) \leq C\|w-a\|\}, \quad (a, z) \in G \times G, \]
  where \( \mathcal{P}SH(G) \) denotes the family of all functions plurisubharmonic on \( G \) (and \( \|\| \) is the Euclidean norm in \( \mathbb{C}^n \)).

• **Lempert function:**
  \[ \tilde{k}^*_G(a, z) := \inf \{m_E(\lambda, \mu) : \lambda, \mu \in E : \exists_{\varphi \in \mathcal{O}(E, G)} : \varphi(\lambda) = a, \varphi(\mu) = z\}
  = \inf \{\mu \in [0, 1) : \exists_{\varphi \in \mathcal{O}(E, G)} : \varphi(0) = a, \varphi(\mu) = z\}, \quad (a, z) \in G \times G. \]

It is well known that
\[ c^*_G = m^{(1)}_G \leq m^{(k)}_G \leq g_G \leq \tilde{k}^*_G, \]
and for any holomorphically contractible family \( (d_G)_{\mathcal{G}} \) we have
\[ c^*_G \leq d_G \leq \tilde{k}^*_G, \quad (1.1.2) \]
i.e. the Möbius family is minimal and the Lempert family is maximal.

Put \( k_G := \tanh^{-1} \tilde{k}^*_G \). The pseudodistance
\[ k_G := \sup \{d : d : G \times G \to \mathbb{R}_+ \text{ is a pseudodistance with } d \leq \tilde{k}^*_G\} \]
is called the *Kobayashi pseudodistance* — cf. [J-P 1993], Ch. 3. Observe that \( (k_G)_{\mathcal{G}} \) satisfies (A’, B).

Notice that one can consider conditions weaker than (B), for instance:

(B’) Condition (1.1.1) holds for every injective holomorphic mapping \( F : G \to D \);

(B’’) Condition (1.1.1) holds for every biholomorphic mapping \( F : G \to D \).

For example:

• **The family** \( (H^*_G)_{\mathcal{G}} \) **of Hahn functions**
  \[ H^*_G(a, z) := \inf \{m_E(\lambda, \mu) : \exists_{\varphi \in \mathcal{O}(E, G)} : \varphi(\lambda) = a, \varphi(\mu) = z\}
  = \inf \{\mu : \exists_{\varphi \in \mathcal{O}(E, G)} : \varphi(0) = a, \varphi(\mu) = z\}, \quad (a, z) \in G \times G, \]
satisfies (A,B’). Obviously, \( \tilde{k}^*_G \leq H^*_G \).

• **The family** \( (b_G)_{\mathcal{G}} \) **of Bergman pseudodistances** (see § 3.5) satisfies (A,B’’).

\(^{(2)}\) For relations between the pluricomplex and classical Green functions in the unit ball see [Car 1997]. For a different pluricomplex Green function see [Ceg 1995], [Edi-Zwo 1998a].
Remark 1.1.2. The notion of the holomorphically contractible family \((d_G)_G\) (Definition 1.1.1) may be extended to the case where \(G\) runs through all connected complex manifolds, complex analytic sets, or even complex spaces. In particular, one can define the Möbius pseudodistance \(c_M^*\), the Lempert function \(k_M^*\) (defined as 1 for pairs of points for which there is no analytic disc passing through them), and the Kobayashi pseudodistance \(k_M\) for arbitrary connected complex analytic set \(M\). The following elementary example points out some new problems appearing in this case.

Let \(M := \{(z,w) \in E^2 : z^2 = w^3\}\) be the Neil parabola. \(M\) is a connected one-dimensional analytic subset of \(E^2\) with \(\text{Reg} M = M = M \setminus \{(0,0)\}\). The set \(M\) has a global bijective holomorphic parametrization

\[
E \ni \lambda \mapsto (\lambda^3, \lambda^2) \in M.
\]

- The mapping \(q := p^{-1}\) is holomorphic on \(M_*\) and continuous on \(M\). Note that \(q(z,w) = z/w, (z,w) \in M_*\), \(q(0,0) = 0\).
- The mapping \(q|M_* : M_* \to E\) is biholomorphic. Thus

\[
c_M^*((a,b),(z,w)) = m_{E_*}(q(a,b),q(z,w)) = m_E(q(a,b),q(z,w)),
\]

\[
k_M^*((a,b),(z,w)) = k_E^*(q(a,b),q(z,w)), \quad (a,b),(z,w) \in M_*.
\]

- For any \(\varphi \in \mathcal{O}(E,M)\) there exists a \(\psi \in \mathcal{O}(E,E)\) such that \(\varphi = p \circ \psi\). Hence

\[
k_M^*((a,b),(z,w)) = m_E(q(a,b),q(z,w)), \quad (a,b),(z,w) \in M_*. \quad \text{(Remark 1.1.2)}
\]

- For any \(f \in \mathcal{O}(M,E)\) the holomorphic function \(h := f \circ p : E \to E\) satisfies \(h'(0) = 0\). Conversely, for any \(h \in \mathcal{O}(E,E)\) with \(h'(0) = 0\) the function \(f := h \circ q\) is holomorphic on \(M\). Hence

\[
c_M^*((a,b),(z,w)) = \sup\{|h(q(z,w))| : h \in \mathcal{O}(E,E), h(q(a,b)) = 0, h'(0) = 0, (a,b),(z,w) \in M_*\}.
\]

It is a little bit surprising that, despite the elementary description, an effective formula for \(c_M^*\) is not known.

One can prove that for any \(\lambda_0 \in E_*\), we have

\[
\sup\{|h| : h \in \mathcal{O}(E,E), h(\lambda_0) = 0, h'(0) = 0\} = \sup\{|B| : B \text{ is a Blaschke product of order } \leq 3, B(\lambda_0) = 0, B'(0) = 0\}.
\]

1.2. Holomorphically contractible families of pseudometrics

Parallel to the category of holomorphically contractible families of functions (in the sense of Definition 1.1.1) one studies holomorphically contractible families of pseudometrics (cf. [J-P 1993], § 4.1).

**Definition 1.2.1.** A family \((d_G)_G\) of \(\mathbb{C}\)-pseudometrics \(\delta_G : G \times \mathbb{C}^n \to \mathbb{R}_+, G \subseteq \mathbb{C}^n\),

\[
\delta_G(a;\lambda X) = |\lambda|\delta_G(a;X), \quad a \in G, \ X \in \mathbb{C}^n, \ \lambda \in \mathbb{C},
\]

\(\text{(1)}\) \(\text{Reg} M\) denotes the set of all regular points of \(M\).

\(\text{(2)}\) \(A_* := A \setminus \{0\} (A \subseteq \mathbb{C}^n), A_*^n := (A_*)^n, \text{ e.g. } E_*, \mathbb{C}_*, (\mathbb{Z}_*)_*^n, \mathbb{C}_*^n\).
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where $G$ runs over all domains $G \subset \mathbb{C}^n$, is said to be **holomorphically contractible** if the following two conditions are satisfied:

(A) $\delta_E(a;X) = \gamma_E(a;X) := \frac{|X|}{|a|^2}$, \quad $a \in E$, $X \in \mathcal{C}$,

(B) for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ and for every holomorphic mapping $F: G \rightarrow D$ we have

$$\delta_D(F(a); F'(a)(X)) \leq \delta_G(a;X), \quad (a, X) \in G \times \mathbb{C}^n. \quad (1.2.3)$$

The following holomorphically contractible families of pseudometrics correspond to the holomorphically contractible families of functions from § 1.1.

- **Carathéodory–Reiffen pseudometric**:

$$\gamma_G(a;X) := \sup\{|f'(a)(X)| : f \in \mathcal{O}(G,E), \ f(a) = 0\}, \quad (a, X) \in G \times \mathbb{C}^n;$$

we have

$$\gamma_G(a;X) = \lim_{\zeta, \zeta' \rightarrow 0 \atop \zeta \neq \zeta'} \frac{1}{|a|} d_G^*_G(a, a + \lambda X) = \lim_{\zeta, \zeta' \rightarrow 0 \atop \zeta \neq \zeta'} C_G^*(\gamma', \gamma'') \|\gamma' - \gamma''\|, \quad (a, X) \in G \times \mathbb{C}^n, \|X\| = 1. \quad (1.2.4)$$

- **Higher order Reiffen pseudometric**:

$$\gamma_G^{(k)}(a;X) := \sup\left\{ \frac{1}{k!} |f^{(k)}(a)(X)|^{1/k} : f \in \mathcal{O}(G,E), \ \text{ord}_a f \geq k \right\}, \quad (a, X) \in G \times \mathbb{C}^n, \ k \in \mathbb{N};$$

we have

$$\gamma_G^{(k)}(a;X) = \lim_{\zeta, \zeta' \rightarrow 0 \atop \zeta \neq \zeta'} \frac{1}{|a|} m_G^{(k)}(a, a + \lambda X), \quad (a, X) \in G \times \mathbb{C}^n,$$

and if $G$ is biholomorphic to a bounded domain, then

$$\gamma_G^{(k)}(a;X) = \lim_{\zeta, \zeta' \rightarrow 0 \atop \zeta \neq \zeta'} \frac{m_G^{(k)}(\gamma', \gamma'')}{\|\gamma' - \gamma''\|}, \quad (a, X) \in G \times \mathbb{C}^n, \|X\| = 1. \quad (1.2.5)$$

- **Azukawa pseudometric**:

$$A_G(a;X) := \liminf_{\zeta, \zeta' \rightarrow 0 \atop \zeta \neq \zeta'} \frac{1}{|a|} g_G(a, a + \lambda X), \quad (a, X) \in G \times \mathbb{C}^n;$$

if $G$ is a bounded hyperconvex domain, then

$$A_G(a;X) = \lim_{\zeta, \zeta' \rightarrow 0 \atop \zeta \neq \zeta'} \frac{g_G(\gamma', \gamma'')}{\|\gamma' - \gamma''\|}, \quad (a, X) \in G \times \mathbb{C}^n, \|X\| = 1,$$

cf. [Zwo 2000c], Corollary 4.4.

- **Kobayashi–Royden pseudometric**:

$$\kappa_G(a;X) := \inf\{ \alpha \geq 0 : \exists \varphi \in \mathcal{O}(E,G) : \varphi(0) = a, \ \alpha \varphi'(0) = X\}, \quad (a, X) \in G \times \mathbb{C}^n;$$
if $G$ is taut, then
$$
\gamma_G(a;X) = \lim_{\lambda \to 0} \frac{1}{|\lambda|} \tilde{k}_G(a, a + \lambda X), \quad (a, X) \in G \times \mathbb{C}^n,
$$
cf. [Pan 1994].

It is well known that $\gamma_G = \gamma_G^{(1)} \leq \gamma_G^{(k)} \leq A_G \leq \varkappa_G$. Moreover, for any holomorphically contractible family of pseudometrics $G$ we have $\gamma_G \leq \delta_G \leq \varkappa_G$ for any $G$. Notice that (cf. [J-P 1993], [Jar-Pfl 1995b]):

- $\gamma_G$ is Lipschitz continuous;
- $\gamma_G^{(k)}$ is upper semicontinuous; if $\gamma_G(a;X) > 0$, $(a, X) \in \mathbb{C} \times (\mathbb{C}^n)_*$, then $\gamma_G^{(k)}$ is continuous (cf. [Nik 2000]); in particular, if $G$ is bounded, then $\gamma_G^{(k)}$ is continuous;
- $A_G$ is upper semicontinuous;
- $\varkappa_G$ is upper semicontinuous; if $G$ is taut, then $\varkappa_G$ is continuous.

Similarly as in the case of contractible functions, one can consider conditions weaker than (B), for example:

(B') Condition (1.2.3) holds for every injective holomorphic mapping $F : G \to D$;
(B'') Condition (1.2.3) holds for every biholomorphic mapping $F : G \to D$.

For example:

- The family $(h_G)_G$ of Hahn pseudometrics
  $$
h_G(a;X) := \inf \{a \geq 0 : \exists \varphi \in \mathcal{O}(E,G) : \varphi \text{ is injective}, \varphi(0) = a, \alpha\varphi'(0) = X\},
  \quad (a, X) \in G \times \mathbb{C}^n,
$$
  fulfills (A,B'). Obviously, $\varkappa_G \leq h_G$.
- The families of Wu and Bergman pseudometrics satisfy (A,B'') (see §§ 1.2.6, 3.5).

Remark 1.2.2. (a) If $G \subset \mathbb{C}$, then $\tilde{k}_G \equiv H_G^* \iff \varkappa_G \equiv h_G \iff G$ is simply connected.
(b) $\tilde{k}_G^* \equiv H_G^*$ and $\varkappa_G \equiv h_G$ for any domain $G \subset \mathbb{C}^n$ with $n \geq 3$ (cf. [Ove 1995]).
(c) Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then (cf. [JarW 2000], [JarW 2001]):
  - if at least one of the domains $D_1, D_2$ is simply connected or biholomorphic to $\mathbb{C}_*$, then $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$ and $H_{D_1 \times D_2}^* \equiv \tilde{k}_{D_1 \times D_2}^*$;
  - otherwise $h_{D_1 \times D_2} \not\equiv \varkappa_{D_1 \times D_2}$ and $H_{D_1 \times D_2}^* \not\equiv \tilde{k}_{D_1 \times D_2}^*$; see also [Choi 1998].

1.2.1. Inner pseudodistances. For a domain $G \subset \mathbb{C}^n$ let $\mathcal{D}(G)$ be the family of all pseudodistances $\rho : G \times G \to \mathbb{R}_+$ such that
$$
\forall_{a \in G} \exists_{M, r > 0} : \rho(z, w) \leq M \|z - w\|, \quad z, w \in B(a, r) \subset G,
$$
where $B(a, r)$ denotes the Euclidean ball with center at $a$ and radius $r$. Notice that for any holomorphically contractible family of pseudodistances $(d_G)_G$ (with the normalization condition (A) or (A')) we have $d_G \in \mathcal{D}(G)$ for any $G$. Let $\mathcal{F}$ be one of the following three families of curves in $\mathbb{C}^n$:

- $\mathcal{F}_{in} :=$ the family of all curves,
- $\mathcal{F}_i :=$ the family of all rectifiable curves (in the Euclidean sense),
- $\mathcal{F}_{ic} :=$ the family of all piecewise $C^1$–curves.
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For any \( \rho \in \mathcal{D}(G) \) we define the inner pseudodistance for \( \rho \) w.r.t. the family \( \mathcal{F} \):

\[
\rho^F(a, z) := \inf \{ L(\rho) : \alpha : [0, 1] \to G, \, \alpha(0) = a, \, \alpha(1) = z, \, \alpha \in \mathcal{F} \}, \quad (a, z) \in G \times G,
\]

where \( L(\rho) \) is the \( \rho \)-length of \( \alpha \):

\[
L(\rho) := \sup \left\{ \sum_{j=1}^{N} \rho(\alpha(t_{j-1}), \alpha(t_j)) : 0 = t_0 < t_1 < \cdots < t_N = 1, \, N \text{ arbitrary} \right\}.
\]

Note that:

- \( \rho^F \in \mathcal{D}(G) \),
- \( \rho^F \geq \rho \),
- \( L(\rho) = L(\rho) \) for any \( \alpha \in \mathcal{F} \),
- \( (\rho^F)^F = \rho^F \) for \( \mathcal{F} \subset \mathcal{G} \),
- \( (\rho^F)^F = \rho^F \),
- \( \rho^F = (\tanh F) \).

We put:

- \( \rho^m := \rho^F \) (cf. [Rin 1961]),
- \( \rho^I := \rho^F \) (cf. [J-P 1993]),
- \( \rho^C := \rho^F \) (cf. [Ven 1989]).

Note that \( \rho \leq \rho^m \leq \rho^I \leq \rho^C \). We say that \( \rho \) is inner if \( \rho = \rho^C \) (in particular, \( \rho = \rho^m = \rho^I = \rho^C \)); see also [Bar 1995].

In particular, we introduce the inner Carathéodory pseudodistance \((c^I_G)\). It is known that:

- \( c^I_G = c^I_G \) for any \( G \);
- \( c^m_G = c^I_G = c^I_G \) if \( G \) is biholomorphic to a bounded domain or \( G \subset \mathbb{C}^1 \) — notice that \( \exists \) in the general case the equality \( c^m_G = c^I_G \) remains still open \( \exists \);
- \( c^I \neq c^I \) — for instance \( c_A \neq c_A \) if \( A \subset \mathbb{C} \) is an annulus (cf. [J-P 1993], Example 2.5.7, see also [Jar-Pfl 1993a]);
- \( n^I_B = p_E = p_E \).

On the other hand, the Kobayashi pseudodistance is obviously inner, i.e. \( k_G = k^m_G = k^I_G = k^I_G \) for any \( G \) — cf. [J-P 1993], Proposition 3.3.1.

If \((d_G)\) is a holomorphically contractible family of pseudodistances (with the normalization condition \((A)\) or \((A')\)), then the families \((d^m_G)\), \((d^I_G)\), \((d^C_G)\) are holomorphically contractible with the normalization condition \((A')\).

1.2.2. Integrated forms. The idea of inner pseudodistances is strictly connected with the idea of integrated forms from differential geometry. More precisely, for a domain \( G \subset \mathbb{C}^n \), let \( \mathcal{M}(G, K) \) \((K \in \{ \mathbb{R}, \mathbb{C} \})\) denote the space of all \( K \)-pseudometrics

\[
\eta : G \times \mathbb{C}^n \to \mathbb{R}_+, \quad \eta(a; tX) = |t| \eta(a; X), \quad (a, X) \in G \times \mathbb{C}^n, \quad t \in K\]

such that

\[
\forall a \in G \exists M > 0 : \eta(z; X) \leq M |X|, \quad z \in B(a, r) \subset G, \quad X \in \mathbb{C}^n.
\]

\((\text{2})\) Notice that so far we have used only \( \mathbb{C} \)-pseudometrics (cf. Definition 1.2.1).
If $\eta \in \mathcal{M}(G, \mathbb{K})$ is Borel measurable (e.g. $\eta \in \{\gamma_G, \gamma^{(k)}_G, A_G, \varkappa_G\}$), then we define the integrated form of $\eta$:

$$(\int \eta)(a, z) := \inf \{L_\eta(\alpha) : \alpha : [0, 1] \rightarrow G, \; \alpha(0) = a, \; \alpha(1) = z, \; \alpha \in \mathcal{F}_G\}, \quad a, z \in G,$$

where $L_\eta(\alpha)$ is the $\eta$–length of $\alpha$:

$$L_\eta(\alpha) := \int_0^1 \eta(\alpha(t); \alpha'(t))dt.$$

One can easily prove that $\int \eta \in \mathcal{D}(G)$ and $(\int \eta)^{ic} = \int \eta$, i.e. $\int \eta$ is always inner.

1.2.3. Buseman pseudometric. Let $h : \mathbb{C}^n \rightarrow \mathbb{R}_+$ be such that:

- $h(\lambda X) = |\lambda| h(X)$, $X \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$,
- there exists a constant $M > 0$ for which $h(X) \leq M \|X\|$, $X \in \mathbb{C}^n$.

Define the Buseman seminorm for $h$

$$\hat{h} := \sup\{q : \; q \text{ is a } \mathbb{C}\text{-seminorm, } q \leq h\};$$

note that $\hat{h}$ is a $\mathbb{C}$–seminorm (in particular, $\hat{h}$ is continuous) and $\hat{h} \leq h$; cf. [J-P 1993], § 4.3).

If $\eta \in \mathcal{M}(G, \mathbb{C})$ (cf. § 1.2.2), then we define the Buseman pseudometric associated to $\eta$,

$$(\hat{\eta})(a; X) := (\eta(a; \cdot)^{\gamma}(X), \quad (a, X) \in G \times \mathbb{C}^n;$$

cf. [J-P 1993], § 4.3. In particular, we define the Kobayashi–Buseman pseudometric $\hat{\varkappa}_G$.

Recall that:

- if $\eta$ is upper semicontinuous, then so is $\hat{\eta}$;
- if $(\delta_G)\hat{\varkappa}$ is a holomorphically contractible family of pseudometrics, then so is $(\hat{\delta}_G)\hat{\varkappa}$.

**Remark 1.2.3.** (a) One can easily prove that if $\eta$ is a continuous metric ($\eta(a; X) > 0$, $(a, X) \in G \times (\mathbb{C}^n)_\ast$), then so is $\hat{\eta}$ (cf. the proof of Proposition 1.2.13(a)).

(b) The following example (due to W. Jarnicki) shows that if $\eta$ is a continuous pseudometric, then $\hat{\eta}$ need not be continuous.

Let $\eta : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{R}_+$, $\eta(z; X) := \max\{0, |X_2| - \|z\| \|X_1\|\}$. Then $\eta$ is a continuous pseudometric. Observe that $\eta(0; X) = |X_2| = \hat{\eta}(0; X)$ and $\hat{\eta}(z; \cdot) \equiv 0$ for $z \neq 0$ (in particular, $\hat{\eta}$ is not continuous). Indeed, for $z \neq 0$ we have $\hat{\eta}(z; (X_1, 0)) \leq \eta(z; (X_1, 0)) = 0$ and

$$\hat{\eta}(z; (0, X_2)) \leq \hat{\eta}(z; (0, 2X_2)) \leq \hat{\eta}(z; (X_2/\|z\|, X_2)) + \hat{\eta}(z; (-X_2/\|z\|, X_2))$$

$$\leq \eta(z; (X_2/\|z\|, X_2)) + \eta(z; (-X_2/\|z\|, X_2)) = 0.$$

1.2.4. Derivatives. It is natural to conjecture that

(*) for any $\rho \in \mathcal{D}(G)$ (cf. § 1.2.1) there exists a Borel measurable pseudometric $\eta = \eta(\rho) \in \mathcal{M}(G, \mathbb{K})$ such that $\rho^{ic} = \int \eta$ and if $(d_G)\hat{\varkappa}$ is a holomorphically contractible family of pseudodistances, then $(\eta(d_G))\hat{\varkappa}$ is a contractible family of $\mathbb{K}$–pseudometrics.
Remark 1.2.4. (a) Recall that $k_G = \int \rho_G = \int \tilde{\rho}_G$, where $\tilde{\rho}_G$ is the Kobayashi–Buseman pseudometric (§1.2.3); cf. [Ven 1996] for a generalization of the formula $k_G = \int \rho_G$ to the case of analytic spaces.

Thus, we can take $\eta(k_G) := \rho_G$ or $\eta(k_G) := \tilde{\rho}_G$.

(b) Notice that in general $\rho_G$ is not determined by $k_G$; there exists a pseudoconvex Hartogs domain $G \subset \mathbb{C}^2$ such that $k_G \equiv 0$ and $\rho_G \not\equiv 0$ (cf. [J-P 1993], Example 3.5.10).

(c) It is known that the problem (*) has a positive solution in the category of so-called $C^1$–pseudodistances, i.e. those pseudodistances $\rho \in \mathcal{D}(G)$ for which the limit

$$(\mathcal{D}\rho)(a; X) = \lim_{\lambda \to 0} \frac{1}{\lambda} \rho(z, z + \lambda Y)$$

exists for all $(a, X) \in G \times \mathbb{C}^n$ and the function

$$(a, X) \in G \times \mathbb{C}^n \mapsto (\mathcal{D}\rho)(a; X)$$

is continuous. If $\rho$ is a $C^1$–pseudodistance, then

$$(\mathcal{D}\rho)(a; X) = \lim_{z' \to a} \frac{\rho(z', z'' \lambda)}{\|z' - z''\|}, \quad (a, X) \in G \times \mathbb{C}^n, \|X\| = 1;$$

$\rho' = \rho^{ic} = \int(\mathcal{D}\rho)$, $\rho'$ is a $C^1$–pseudodistance, and $\mathcal{D}\rho' = \mathcal{D}\rho$ (cf. [J-P 1993], Proposition 4.3.9).

In particular, since $c_G$ is a $C^1$–pseudodistance, we have $\eta(c_G) = \eta(c'_G) = \gamma_G$.

(d) M. Kobayashi proved in [KobM 2000] that if $G$ is taut, then $k_G$ is a $C^1$–pseudodistance.

In the case $K = \mathbb{C}$ the problem (*) seems to be open (cf. [J-P 1993], the remark after Theorem 4.3.10). Surprisingly, in the case $K = \mathbb{R}$ (*) has the following complete solution. For $\rho \in \mathcal{D}(G)$ define

$$(\mathcal{D}\rho)(a; X) := \limsup_{t \to 0} \frac{1}{|t|} \rho(a, a + tX), \quad (a, X) \in G \times \mathbb{C}^n;$$

cf. [Ven 1989]. We say that $\mathcal{D}\rho$ is the weak derivative of $\rho$. One can prove that:

- $\mathcal{D}\rho \in M(G, \mathbb{R})$, $\mathcal{D}\rho$ is Borel measurable.
- $$(\mathcal{D}\rho)(a; X) = \limsup_{Y \to X} \frac{1}{|Y|} \rho(a, a + tY), \quad (a, X) \in G \times \mathbb{C}^n,$$

$$(\mathcal{D}\rho)(a; X) = \limsup_{z \to a} \frac{\rho(a, z)}{|a - z|}, \quad (a, X) \in G \times \mathbb{C}^n, \|X\| = 1.$$

- $L_\rho(\alpha) = L_{\mathcal{D}\rho}(\alpha)$ for any piecewise $C^1$–curve $\alpha : [0, 1] \to G$. In particular, $\rho^{ic} = \int(\mathcal{D}\rho)$.
- If $(d_G)_G$ is a holomorphically contractible family of pseudodistances, then $(\mathcal{D}(d_G))_G$ is a holomorphically contractible family of $\mathbb{R}$–pseudometrics.
- $\int(\mathcal{D}k_G) = k_G$. 

1.2. Holomorphically contractible families of pseudometrics

1.2.5. Complex geodesics. Recall that a holomorphic mapping \( \varphi : E \rightarrow G \) (\( G \) is a domain in \( \mathbb{C}^n \)) is called a complex geodesic if \( c^G_\varphi(\varphi(\lambda'), \varphi(\lambda'')) = m_E(\lambda', \lambda'') \) for any \( \lambda', \lambda'' \in E \).

Let \( (d\varphi)_G \) be a holomorphically contractible family of functions. Fix a domain \( G \subset \mathbb{C}^n \) and let \( z'_0, z''_0 \in G \), \( z'_0 \neq z''_0 \). We say that \( \varphi \in \mathcal{O}(E, G) \) is a \( d\varphi \)-geodesic for \( (z'_0, z''_0) \) if there exist \( \lambda'_0, \lambda''_0 \in E \) such that \( \lambda'_0 = \varphi(\lambda'_0) \), \( z''_0 = \varphi(\lambda''_0) \), and \( d\varphi(z'_0, z''_0) = m_E(\lambda'_0, \lambda''_0) \).

If \( \varphi \) is a \( d\varphi \)-geodesic for \( (z'_0, z''_0) \), then \( d\varphi(z'_0, z''_0) = \tilde{k}^G_{\varphi}(z'_0, z''_0) \). Obviously, any complex geodesic is a \( c^G_{\varphi'} \)-geodesic for \( (z'_0, z''_0) \) with arbitrary \( z'_0, z''_0 \in \varphi(E) \), \( z'_0 \neq z''_0 \).

Let \( (\delta_{\varphi})_G \) be a holomorphically contractible family of pseudometrics. Let \( z_0 \in G \), \( X_0 \in \mathbb{C}^n \). We say that \( \varphi \in \mathcal{O}(E, G) \) is a \( \delta_{\varphi} \)-geodesic for \( (z_0, X_0) \) if there exist \( \lambda_0 \in E \), \( \alpha_0 \in \mathbb{C} \) such that \( z_0 = \varphi(\lambda_0) \), \( X_0 = \alpha_0\varphi'(\lambda_0) \), and \( \delta_{\varphi}(z_0; X_0) = \gamma_E(\lambda_0; \alpha_0) \). If \( \varphi \) is a \( \delta_{\varphi} \)-geodesic for \( (z_0, X_0) \), then \( \delta_{\varphi}(z_0; X_0) = \kappa_{\varphi}(z_0; X_0) \).

**Proposition 1.2.5** ([J-P 1993], Proposition 8.1.3). For a mapping \( \varphi \in \mathcal{O}(E, G) \) the following conditions are equivalent:

1. \( \exists \lambda'_0, \lambda''_0 \in E : c^G_{\varphi}(\varphi(\lambda'_0), \varphi(\lambda''_0)) = m_E(\lambda'_0, \lambda''_0) \), i.e. \( \varphi \) is a complex \( c^G_{\varphi} \)-geodesic for \( (\lambda'_0 \neq \lambda''_0) \);
2. \( \forall \lambda, \lambda' \in E : c^G_{\varphi}(\varphi(\lambda), \varphi(\lambda')) = m_E(\lambda, \lambda') \), i.e. \( \varphi \) is a complex geodesic;
3. \( \forall \lambda \in E : \gamma(\varphi(\lambda)); \varphi'(\lambda) = \gamma_E(\lambda; 1) \), i.e. \( \varphi \) is a complex \( \gamma_E \)-geodesic for any pair \( (\varphi(\lambda), \varphi'(\lambda)) \);
4. \( \exists \alpha_0 \in \mathbb{C} : \gamma(\varphi(\lambda_0)); \varphi'(\lambda_0) = \gamma_E(\lambda_0; 1) \), i.e. \( \varphi \) is a complex \( \gamma_E \)-geodesic for \( (\varphi(\lambda_0), \varphi'(\lambda_0)) \).

Consequently, any complex \( c^G_{\varphi} \)- or \( \gamma_E \)-geodesic \( \varphi \) is a complex geodesic. Moreover, \( \varphi \) is injective, proper, and regular. In particular, \( \varphi(E) \) is a 1-dimensional complex submanifold of \( G \).

**Proposition 1.2.6** ([J-P 1993], Proposition 8.1.5). Let \( G \subset \mathbb{C}^n \) be a taut domain. Then the following conditions are equivalent:

1. \( c^G_{\varphi} = \tilde{k}^G_{\varphi} \) and \( \gamma_{\varphi} = \kappa_{\varphi} \) \(^{(6)}\);
2. \( c^G_{\varphi} = \tilde{k}^G_{\varphi} \);
3. for any \( z'_0, z''_0 \in G \), \( z'_0 \neq z''_0 \), there exist \( \varphi \in \mathcal{O}(E, G) \) and \( f \in \mathcal{O}(G, E) \) such that \( z'_0, z''_0 \in \varphi(E) \) and \( f \circ \varphi = \text{id}_E \);
4. for any \( z'_0, z''_0 \in G \) there exist a holomorphic embedding \( \varphi : E \rightarrow G \) and a holomorphic retraction \( r : G \rightarrow \varphi(E) \) such that \( z'_0, z''_0 \in \varphi(E) \).

Moreover, any holomorphic mapping \( \varphi : E \rightarrow G \) satisfying (iii) or (iv) is a complex geodesic. Conversely, for any complex geodesic \( \varphi \) there exists \( f \) (resp. \( r \)) such that (iii) (resp. (iv)) is fulfilled.

Recently Proposition 1.2.5 was generalized in the following way in [EHMM 2003], Corollary 9.

**Proposition 1.2.7.** A holomorphic mapping \( \varphi : E \rightarrow G \) (\( G \) is a domain in \( \mathbb{C}^n \)) is a complex geodesic iff there exist \( \lambda'_0, \lambda''_0 \in E \), \( \lambda'_0 \neq \lambda''_0 \), such that \( c^G_{\varphi}(\varphi(\lambda'_0), \varphi(\lambda''_0)) = p_E(\lambda'_0, \lambda''_0) \).

\(^{(6)}\) For example, \( G \) is a convex domain — cf. Lempert Theorem 8.2.1 in [J-P 1993].
Proof. (Here we present a direct proof independent of [EHHM 2003].) Using a suitable automorphism of $E$, we may assume that $\lambda'_0 = 0$ and $\lambda''_0 =: t_0 \in (0,1)$. Recall that $p_E' = p_E$. Hence, for any $t \in [0,t_0]$, we have

$$p_E(0,t_0) = p_E(0,t) + p_E(t,t_0) \geq c_G'(\varphi(0),\varphi(t)) + c_G'(\varphi(t),\varphi(t_0)) \geq c_G'(\varphi(0),\varphi(t_0)) = p_E(0,t_0).$$

Consequently, $c_G'(\varphi(0),\varphi(t)) = p_E(0,t)$ for any $t \in [0,t_0]$. Let $t_k \to 0$ be such that

$$\frac{\varphi(t_k) - \varphi(0)}{\|\varphi(t_k) - \varphi(0)\|} \to X_0 \in \partial B_n$$

(observe that $X_0 = \varphi'(0)/\|\varphi'(0)\|$ if $\varphi'(0) \neq 0$). Recall that $\mathcal{D}c_G' = \gamma_G$. We get

$$1 = \gamma_E(0;1) = \lim_{k \to \infty} \frac{p_E(0,t_k)}{t_k} = \lim_{k \to \infty} \frac{c_G'(\varphi(0),\varphi(t_k))}{t_k} = \gamma_G(\varphi(0);X_0)\|\varphi'(0)\|.$$ 

Hence $\varphi'(0) \neq 0$ and $1 = \gamma_G(\varphi(0);\varphi'(0))$ which, by Proposition 1.2.5, implies that $\varphi$ is a complex geodesic.

Remark 1.2.8. (a) Complex geodesics were recently studied by many authors. For instance:

- in [Jar-Pfl 1995a] for convex complex ellipsoids

$$E_p := \{(z_1,\ldots,z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\},$$

$$p = (p_1,\ldots,p_n), \ p_j \geq 1/2, \ j = 1,\ldots,n \ (n \geq 2); \quad (\dagger)$$

- in [Zwo 1997] to prove the following result showing that in the category of complex ellipsoids the symmetry of the pluripolar Green function $g_{E_p}$ is a very rare phenomenon:

Theorem. For a complex ellipsoid $E_p$ the following conditions are equivalent:

(i) $k_{E_p}(\lambda_1 b,\lambda_2 b) = p_E(\lambda_1,\lambda_2), \ b \in \partial E_p, \ \lambda_1,\lambda_2 \in E$;

(ii) $g_{E_p}(\lambda b,0) = g_{E_p}(0,\lambda b), \ b \in \partial E_p, \ \lambda \in E$;

(iii) $g_{E_p}$ is symmetric;

(iv) $E_p$ is convex;

- in [Vis 1999a], [Vis 1999b], [Vis 1999c] for some classes of convex Reinhardt domains;

- in [Pfl-You 2003] for the so-called minimal ball

$$M_n := \{z = (z_1,\ldots,z_n) \in \mathbb{C}^n : \|z\|^2 + |z_1^2 + \cdots + z_n^2| < 1\}$$

(which will be studied in § 3.1).
(b) Consider the following general problem: Given a bounded convex balanced domain $G \subset \mathbb{C}^n$ ($n \geq 2$) with Minkowski function $h_G$ \((^{(8)}\) \(^{(9)}\)), find conditions on $a, b \in G$ and $r, R \in (0, 1)$, under which the Carathéodory ball $B_{c_G^0}(a, r) := \{ z \in G : c_G^0(a, z) < r \}$ coincides with the norm ball $B_{h_G}(b, R) := \{ z \in \mathbb{C}^n : h_G(z - b) < R \}$ \(^{(10)}\). Since $c_G^0(0, \cdot) = h_G(\cdot)$, we always have

$$B_{c_G^0}(0, r) = B_{h_G}(0, r), \quad r \in (0, 1).$$

In the case where

$$G = \mathbb{E}_{p, \alpha} := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : 2\alpha |z_1|^{p_1} |z_2|^{p_2} + \sum_{j=1}^{n} |z_j|^{2p_j} < 1 \},$$

the problem was studied in:

- [Sch 1993] (the case $n = 2$, $\alpha = 0$, $p_1 = p_2 = 1$),
- [Sre 1995], [Zwo 1995] (the case $\alpha = 0$, $p_1 = \cdots = p_n = 1$),
- [Sch-Sre 1996] (the case $\alpha = 0$, $1 < p_1 = \cdots = p_n \notin \mathbb{N}$),
- [Zwo 1996], [Zwo 2000b] (the case $\alpha = 0$),
- [Vis 1999a] (the general case).

The methods introduced by W. Zwonek and developed by B. Visintin are based on complex geodesic. The most general result is the following theorem from [Vis 1999a].

**Theorem.** Assume that $\alpha \geq 0$ and $p \in \mathbb{R}_{>0}^n$ are such that $\mathbb{E}_{p, \alpha}$ is convex. Then

$$B_{\mathbb{E}_{p, \alpha}}(a, r) = B_{\mathbb{E}_{p, \alpha}}(b, R)$$

for some $a, b \in \mathbb{E}_{p, \alpha}$, $a \neq 0$, $r, R \in (0, 1)$ iff $\alpha = 0$, \{ $j \in \{ 1, \ldots, n \} : a_j \neq 0$ \} = \{ $j_0$ \}, $p_{j_0} = 1$, and $p_j = 1/2$ for all $j \neq j_0$.

**Remark 1.2.9.** Let $G \subset \mathbb{C}^n$ be a domain and let $(z_0, X_0) \in G \times \mathbb{C}^n$. Recall that a mapping $\varphi \in \mathcal{O}(E, G)$ is called a $\mathcal{K}_G$–geodesic for $(z_0, X_0)$ if there are a $\lambda_0 \in E$ and an $\alpha_0 \in \mathbb{C}$ such that $\varphi(\lambda_0) = z_0$ and $\mathcal{K}_G(z_0; X_0) = \gamma_G(\lambda_0; \alpha_0)$ (see Chapter VIII in [J-P 1993]). If $G$ is taut, then for any pair $(z_0, X_0) \in G \times \mathbb{C}^n$ there is a $\mathcal{K}_G$–geodesic for $(z_0, X_0)$. If $G$ is even convex, then any $\mathcal{K}_G$–geodesic is a complex geodesic.

Now let $G = \mathbb{E}_p$, where $p = (p_1, \ldots, p_n)$ with $p_j > 0$. Observe that $\mathbb{E}_p$ is not necessarily convex. Fix a pair $(z_0, X_0) \in \mathbb{E}_p \times (\mathbb{C}^n)_*$. Then any $\mathcal{K}_{\mathbb{E}_p}$–geodesic $\varphi$ for $(z_0, X_0)$,

\(^{(8)}\) Observe the difference between the Hahn pseudometric $h_G : G \times \mathbb{C}^n \longrightarrow \mathbb{R}_+$ and the Minkowski function $h_G : \mathbb{C}^n \longrightarrow \mathbb{R}_+$; since the Hahn pseudometric will be not used in the sequel, there will be no confusion for the reader.

\(^{(9)}\) Notice that under our assumptions $h_G$ is a complex norm.

\(^{(10)}\) Recall that in the case of the unit disc we have

$$B_{m_k}(a, r) = \mathbb{B} \left( \frac{a(1 - r^2)}{1 - r^2 |a|^2} \cdot \frac{r(1 - |a|^2)}{1 - r^2 |a|^2}, \quad a \in E, \ r \in (0, 1). \right)$$

\(^{(11)}\) $A_{>0} := \{ x \in A : x > 0 \} (A \subset \mathbb{R})$, $A_{>0}^n := (A_{>0})^n$, e.g. $\mathbb{R}_{>0}, \mathbb{R}_{>0}^n$. Observe that $\mathbb{E}_{p, 0} = \mathbb{E}_p$. 

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**1.2. Holomorphically contractible families of pseudometrics**

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where \( \varphi_j \not\equiv 0, \ j = 1, \ldots, n, \) is necessarily of the following form (see [Pfl-Zwo 1996])
\[
\varphi_j(\lambda) = B_j(\lambda) \left( a_j \frac{1 - \overline{\alpha}_j \lambda}{1 - \overline{\alpha}_0 \lambda} \right)^{\frac{1}{p_j}}, \quad j = 1, \ldots, n,
\]
where \( B_j \) is a Blaschke product and the complex numbers \( a_j, \alpha_j \) fulfill the following conditions
- \( a_j \in \mathbb{C}, \ \alpha_j \in E, \ j = 1, \ldots, n, \ \alpha_0 \in E, \)
- \( 1 + |\alpha_0|^2 = \sum_{j=1}^n |a_j|^2(1 + |\alpha_j|^2), \)
- \( \alpha_0 = \sum_{j=1}^n |a_j|^2 \alpha_j. \)
Moreover, if \( p_j \geq \frac{1}{2}, \) then \( B_j \equiv 1 \) or \( B_j(\lambda) = \frac{\lambda - \alpha_j}{1 - \alpha_j \lambda} \) with \( |\alpha_j| < 1. \)

Additionally, if \( \alpha_j \in E \) for all \( j = 1, \ldots, n, \) then either \( B_j \equiv 1 \) or \( B_j(\lambda) = \frac{\lambda - \alpha_j}{1 - \alpha_j \lambda} \) for all \( j = 1, \ldots, n. \)

Using this result, the Kobayashi metric for the non convex domain \( E_{(1,m)}, 0 < m < \frac{1}{2}, \) is obtained.

First observe that the following mappings
\[
E_{(1,m)} \ni z \mapsto \left( \frac{z_1 - a}{1 - \overline{a} z_1}, \frac{e^{i\theta}(1 - |a|^2)^{1/(2m)} z_2}{(1 - \overline{a} z_1)^{1/m}} \right) \in E_{(1,m)}, \quad a \in E, \ \theta \in \mathbb{R},
\]
are automorphisms. Therefore, to know \( \tau_{E_{(1,m)}} \), it suffices to calculate \( \tau_{E_{(1,m)}}((0,b); \cdot), \)
\( b \geq 0. \) The simple part is given by the following formulas
- \( \tau_{E_{(1,m)}}((0,0); X) = h_{E_{(1,m)}}(X), \) where \( h_{E_{(1,m)}} \) denotes the Minkowski function of \( E_{(1,m)}, \ X \in \mathbb{C}^2; \)
- \( \tau_{E_{(1,m)}}((0,b); X) = \frac{|X_2|}{b}, \ b > 0, X_1 = 0; \)
- \( \tau_{E_{(1,m)}}((0,0); X) = \frac{|X_1|}{(1 - b^2)^{1/2}}, \ b > 0, X_2 = 0. \)

To discuss the remaining case \( (b > 0 \text{ and without loss of generality}) X = (X_1, 1) \in \mathbb{C}^2, \) we put
\[
\nu := \nu(m, b, X) := \left( \frac{|b| |X_1|}{m} \right)^2.
\]
Moreover, in the case \( \nu \leq \frac{1}{4m(1-m)} \) set
\[
t := t(m, b, X) := \frac{2m^2 \nu}{1 + 2m(m - 1)\nu + \sqrt{1 + 4m(m - 1)\nu}}.
\]
Observe that then the following function
\[
\xi^{2m} - t \xi^{2m-2} - (1 - t)b^{2m}, \quad \xi \in \mathbb{R},
\]
has exactly one zero \( x = x(m, b, X) \) in the interval \((0,1). \) Now we are able to give the remaining formulas.

**Theorem.** Let \( m \in (0, \frac{1}{2}), \ b > 0, X = (X_1, 1) \in \mathbb{C}^2, \nu = \nu(b, m, X), \) and \( x = x(b, m, X) \)
if \( \nu \leq \frac{1}{4m(1-m)}. \) Then:
- if \( \nu \leq 1, \) then
  \[
  \tau_{E_{(1,m)}}((0,0); X) = \frac{m}{b} \frac{x^{2m-1}}{(1-m)x^{2m} + mx^{2m-2} - b^2m} =: \tau_1(\nu);
  \]
• if \( \nu \geq \frac{1}{4m(1-m)} \), then
\[
\kappa_{E(1,m)}((0,b);X) = \frac{m}{b} \sqrt{(1 - 2^m)b^m + b^m} =: \kappa_2(\nu);
\]

• if \( 1 < \nu < \frac{1}{4m(1-m)} \), then
\[
\kappa_{E(1,m)}((0,b);X) = \max\{\kappa_1(\nu), \kappa_2(\nu)\}.
\]

The minimum in the last formula is equal to \( \kappa_1(\nu) \) for \( \nu \leq \nu_0 \) and equal to \( \kappa_2(\nu) \) for \( \nu > \nu_0 \), where
\[
\nu_0 := \frac{t_0}{(t_0(1 - m) + m)^2}, \quad t_0 := \frac{x_0^2 - b^{2m}}{x_0^{2m-2} - b^{2m}}
\]
and \( x_0 \) is the only solution in the interval \( (0, 1) \) of the following equation
\[
\xi^{4m-2}(-1 - 2m + 2m^2 + b^{2m}) + \xi^{2m}(1 + 2m)b^{2m}
+ \xi^{4m-2}(1 + (2m - 1)b^{2m}) - (1 - m)^2\xi^{4m} - m^2\xi^{4m-2} - b^{2m} = 0
\]

It turns out that there is a mapping \( \varphi \in O(E,E(1,m)) \) of the form (1.2.4) which is not a \( \kappa_{E(1,m)} \)-geodesic for \( (\varphi(0), \varphi'(0)) \). Moreover, for a \( b > 0 \) such that \( (0,b) \in G \) the function \( \kappa_{E(1,m)}((0,b);(\cdot, 1)) \) is not differentiable on \( C \).

1.2.6. Wu pseudometric. The Wu pseudometric has been introduced by H. Wu in [Wu 1993] (and [Wu]). Various properties of the Wu pseudometric have been studied in [Che-Kim 1996], [Che-Kim 1997], [Kim 1998], [Che-Kim 2003], [Jar-Pfl 2003a], [Juc 2002].

Following [Jar-Pfl 2003a], let us formulate the definition of the Wu pseudometric in an abstract setting. Let \( h : C^n \rightarrow \mathbb{R}_+ \) be a \( C \)-seminorm. Put:
\[
I = I(h) := \{X \in C^n : h(X) < 1\} \quad (I \text{ is convex}),
\]
\[
V = V(h) := \{X \in C^n : h(X) = 0\} \subset I \quad (V \text{ is a vector subspace of } C^n),
\]
\[
U = U(h) := \text{the orthogonal complement of } V \text{ with respect to the standard Hermitian scalar product } (z, w) = \sum_{j=1}^n z_j \overline{w_j} \text{ in } C^n,
\]
\[
I_0 := I \cap U, \quad h_0 := h|_I \quad (h_0 \text{ is a norm on } U, \ I = I_0 + V).
\]

For any pseudo–Hermitian scalar product \( s : C^n \times C^n \rightarrow C \) (12), let
\[
q_s(X) := \sqrt{s(X,X)}, \ X \in C^n, \quad E(s) := \{X \in C^n : q_s(X) < 1\}.
\]
Consider the family \( F \) of all pseudo–Hermitian scalar products \( s : C^n \times C^n \rightarrow C \) such that \( I \subset E(s) \), equivalently, \( q_s \leq h \). In particular,
\[
V \subset I = I_0 + V \subset E(s) = E(s_0) + V;
\]

(12) That is,
• \( s(\cdot, w) : C^n \rightarrow C \) is \( C \)-linear for any \( w \in C^n \),
• \( s(z, w) = s(w, z) \) for any \( z, w \in C^n \),
• \( s(z, z) \geq 0 \) for any \( z \in C^n \) (if \( s(z, z) > 0 \) for any \( z \in \{C^n\} \), then \( s \) is a Hermitian scalar product).
where \( s_0 := s|_{U \times U} \) (note that \( \mathbb{E}(s_0) = \mathbb{E}(s) \cap U \)). Let \( \text{Vol}(s_0) \) denote the volume of \( \mathbb{E}(s_0) \) with respect to the Lebesgue measure of \( U \). Since \( I_0 \) is bounded, there exists an \( s \in \mathcal{F} \) with \( \text{Vol}(s_0) < +\infty \). Observe that for any basis \( e = (e_1, \ldots, e_m) \) of \( U \) (\( m := \dim_{\mathbb{C}} U \)) we have

\[
\text{Vol}(s_0) = \frac{C(e)}{\det S},
\]

where \( C(e) > 0 \) is a constant (independent of \( s \)) and \( S = S(s_0) \) denotes the matrix representation of \( s_0 \) in the basis \( e \), i.e. \( S_{j,k} := s(e_j, e_k) \), \( j, k = 1, \ldots, m \). In particular, if \( U = \mathbb{C}^m \times \{ 0 \}^{n-m} \) and \( e = (e_1, \ldots, e_m) \) is the canonical basis, then \( C(e) = A_{2m}(\mathbb{B}_m) \), where \( A_{2m} \) denotes the Lebesgue measure in \( \mathbb{C}^m \). We are interested in finding an \( s \in \mathcal{F} \), for which \( \text{Vol}(s_0) \) is minimal, equivalently, \( \det S(s_0) \) is maximal.

Observe that, if \( s \) has the above property with respect to \( h \) (i.e. the volume of \( \mathbb{E}(s_0) \) is minimal), then, for any \( \mathbb{C} \)-linear isomorphism \( L : \mathbb{C}^n \rightarrow \mathbb{C}^n \), the scalar product

\[
\mathbb{C}^n \times \mathbb{C}^n \ni (X, Y) \xrightarrow{L(s)} s(L(X), L(Y)) \in \mathbb{C}
\]

has the analogous property with respect to \( h \circ L \). In particular, this permits us to reduce the situation to the case where \( U = \mathbb{C}^m \times \{ 0 \}^{n-m} \) and next to assume that \( m = n \) (by restricting all the above objects to \( \mathbb{C}^n \approx \mathbb{C}^m \times \{ 0 \}^{n-m} \)).

**Lemma 1.2.10.** There exists exactly one element \( s^h \in \mathcal{F} \) such that

\[
\text{Vol}(s^h) = \min \{ \text{Vol}(s_0) : s \in \mathcal{F} \} < +\infty.
\]

**Proof.** ([Wu], [Wu 1993]) We may assume \( U(h) = \mathbb{C}^n \). First we prove that the set \( \mathcal{F} \) is compact. It is clear that \( \mathcal{F} \) is closed. To prove that \( \mathcal{F} \) is bounded, observe that

\[
|s(e_j, e_k)| \leq \sqrt{s(e_j, e_j)s(e_k, e_k)} = q_s(e_j)q_s(e_k) \leq h(e_j)h(e_k), \quad s \in \mathcal{F}, \quad j, k = 1, \ldots, n,
\]

where \( e_1, \ldots, e_n \) is the canonical basis in \( \mathbb{C}^n \). Consequently, the entries of the matrix \( S(s) \) are bounded (by a constant independent of \( s \)).

Recall that

\[
\text{Vol}(s) = \frac{A_{2n}(\mathbb{B}_n)}{\det S(s)},
\]

Now, using compactness of \( \mathcal{F} \), we see that there exists an \( s^h \in \mathcal{F} \) such that

\[
\text{Vol}(s^h) = \min \{ \text{Vol}(s) : s \in \mathcal{F} \} < +\infty.
\]

It remains to show that \( s^h \) is uniquely determined. Suppose that \( s', s'' \in \mathcal{F} \), \( s' \neq s'' \), are both minimal and let \( S', S'' \) denote the matrix representation of \( s', s'' \), respectively. We know that \( \mu := \det S' = \det S'' \) is maximal (with respect to any basis \( (e_1, \ldots, e_n) \)) in the class \( \mathcal{F} \). Take a basis \( e_1, \ldots, e_n \) such that the matrix \( A := S''(S')^{-1} \) is diagonal and let \( d_1, \ldots, d_n \) be the diagonal elements. Note that \( 1 = \det A = d_1 \cdots d_n \) and that for at least one \( j \in \{ 1, \ldots, n \} \) we have \( d_j \neq 1 \). Let \( s := \frac{1}{2}(s' + s'') \). Then \( s \in \mathcal{F} \). Let \( S = S(s) \) be the matrix representation of \( s \). We have

\[
\det S = \frac{1}{2^n} \det(S' + S'') = \frac{1}{2^n} \det(I_n + A) \det S' \]

\[
= \frac{1 + d_1}{2} \cdots \frac{1 + d_n}{2} \mu > \sqrt{d_1 \cdots d_n} \mu = \mu;
\]
contradiction ($\mathbb{1}_n$ denotes the unit matrix).

Put $s^h := m \cdot s^n (m := \dim U(h))$, $\mathcal{W}h := q \mathcal{W}$ ($\mathcal{W}h(X) = \sqrt{m}s^n(X,X)$, $X \in \mathbb{C}^n$). Obviously, $\mathcal{W}h \leq \sqrt{m}h$ and $\mathcal{W}h = \sqrt{m}h$ if $h = q_\ast$ for some pseudo-Hermitian scalar product $s$. For instance, $\mathcal{W}|| \mathcal{W}|| = \sqrt{\mathcal{W}|| ||}$, where $\mathcal{W}|| ||$ is the Euclidean norm in $\mathbb{C}^n$. Moreover, $\mathcal{W}(\mathcal{W}h) = \sqrt{m}\mathcal{W}h$.

**Remark 1.2.11.** Assume that $U(h) = \mathbb{C}^n$. Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a $\mathbb{C}$–linear isomorphism such that $|\det L| = 1$ and $h \circ L = h$. Then $\text{Vol}(s^h) = \text{Vol}(L(s^h))$ and hence $s^h = L(s^h)$, i.e. $s^h(X,Y) = s^h(L(X),L(Y))$, $X,Y \in \mathbb{C}^n$.

**Proposition 1.2.12.** (a) $h \leq \mathcal{W}h \leq \sqrt{m}h$.

(b) If $h(X) := \max\{h_1(X_1), h_2(X_2)\}$, $X = (X_1, X_2) \in \mathbb{C}^n \times \mathbb{C}^n$, then

$$\hat{s}^h(X,Y) = \hat{s}^{h_1}(X_1, Y_1) + \hat{s}^{h_2}(X_2, Y_2), \quad X = (X_1, X_2), \quad Y = (Y_1, Y_2) \in \mathbb{C}^n \times \mathbb{C}^n.$$ 

In particular,

$$\mathcal{W}h(X) = \left((\mathcal{W}h_1(X_1))^2 + (\mathcal{W}h_2(X_2))^2\right)^{1/2}, \quad X = (X_1, X_2) \in \mathbb{C}^n \times \mathbb{C}^n.$$ 

**Proof.** ([Wu], [Wu 1993]) (a) Using a suitable $\mathbb{C}$–linear isomorphism we may reduce the situation to the case where:

- $U = \mathbb{C}^n$,
- $s^n(X,Y) = (X,Y)$, $X, Y \in \mathbb{C}^n$,
- $\min\{|X| : h(X) = 1\} = \|X\| = a > 0$, $X_\ast = (0, \ldots, 0, a) \in \partial I$; in particular, since $I$ is a balanced convex domain, $I \subset \{(X', X_m) \in \mathbb{C}^{n-1} \times \mathbb{C} : |X_m| < a\}$.

We only need to show that $a \geq 1/\sqrt{n}$. Suppose that $a < 1/\sqrt{n}$ and let $0 < b < 1$ be such that $a^2 + b^2 = 1$. Put $c := a/b$. Note that $(n-1)c^2 < 1$. Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the $\mathbb{C}$–linear isomorphism

$$L(X) := (c\sqrt{n} - 1)X', X_n), \quad X = (X', X_n) \in \mathbb{C}^{n-1} \times \mathbb{C}.$$ 

Obviously, $s^{h\circ L^{-1}} = L^{-1}(s^h)$, so

$$\text{Vol}(s^{h\circ L^{-1}}) = A_{2n}(B_n) |\det L|^2 = A_{2n}(B_n)(c\sqrt{n} - 1)^{2(n-1)}.$$ 

On the other hand, $L(I) \subset \mathbb{B}(a\sqrt{n}) \subset \mathbb{C}^n$. Indeed, for $X = (X', X_n)$ we have

$$\|L(X')\|^2 = (n-1)c^2\|X'\|^2 + |X_n|^2 = (n-1)c^2\|X\|^2 + (1 - (n-1)c^2)|X_n|^2 < (n-1)c^2 + (1 - (n-1)c^2)a^2 = a^2 + (1 - a^2)(n-1)(a^2/b^2) = na^2 < 1.$$ 

Consequently, $\text{Vol}(s^{h\circ L^{-1}}) \leq A_{2n}(B_n)(a\sqrt{n})^{2n}$. Thus, using the above inequality, we get

$$a\sqrt{n} - 1)/(b)^{2(n-1)} \leq (a\sqrt{n})^{2n}.$$ 

Put $f(t) := t(1-t)^{n-1}$, $0 \leq t \leq 1$. Then

$$f\left(a^2\right) = a^2(1 - a^2)^{n-1} \geq \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} = f(1/n);$$

contradiction (because $f$ is strictly increasing in the interval $[0,1/n]$ and $a^2 < 1/n$).
(b) We may assume that \( U(h_j) = \mathbb{C}^{n_j}, \ j = 1, 2. \) Put
\[
s_\ast(X, Y) := \frac{n_1}{n_1 + n_2}s^{h_1}(X_1, Y_1) + \frac{n_2}{n_1 + n_2}s^{h_2}(X_2, Y_2),
\]
\( X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}. \)

We only need to prove that \( \det S(s^h) = \det S(s_\ast) \) (all matrix representations are taken in the canonical bases of \( \mathbb{C}^{n_1} \) and \( \mathbb{C}^{n_2} \), respectively). Let \( s := s^h. \) Since
\[
I(h) = I(h_1) \times I(h_2) \subset \mathbb{E}(s_\ast),
\]
we get \( \det S(s) \geq \det S(s_\ast). \)

Let \( L : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \rightarrow \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \) be the isomorphism of the form \( L(X_1, X_2) := (X_1, -X_2). \) Then \( h \circ L = h \) and, consequently, \( s = L(s) \) (Remark 1.2.11), i.e.
\[
s(X, Y) = s(L(X), L(Y)), \quad X, Y \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.
\]
Hence \( s((X_1, X_2), (Y_1, Y_2)) = 0 \) if \( (X_2 = 0 \) and \( Y_1 = 0) \) or \( (X_1 = 0 \) and \( Y_2 = 0). \) Indeed,
\[
s((X_1, 0), (0, Y_2)) = s(L(X_1, 0), L(0, Y_2)) = s((X_1, 0), (0, -Y_2)) = s((X_1, 0), (0, Y_2)) = -s((X_1, 0), (0, Y_2)).
\]
Consequently,
\[
s(X, Y) = s_1(X_1, Y_1) + s_2(X_2, Y_2), \quad X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2},
\]
where \( s_j \) is a Hermitian scalar product in \( \mathbb{C}^{n_j}, \ j = 1, 2. \) It is clear that \( I(h_j) \subset \mathbb{E}(s_j), \ j = 1, 2. \) Let \( c_j \leq 1 \) be the minimal number such that \( I(h_j) \subset \mathbb{E}(c_j^{-2}s_j), \ j = 1, 2. \) Assume that \( X_0^j \in \partial I(h_j) \) is such that \( s_j(X_0^j, X_0^j) = c_j^2, \ j = 1, 2. \) In particular, \( q_s(X_0^1, X_0^2) \leq 1, \) so \( c_1^2 + c_2^2 \leq 1. \) We have
\[
\det S(s^h) \geq c_j^{-2n_j} \det S(s_j), \quad j = 1, 2,
\]
and, therefore,
\[
\det S(s) = \det S(s_1) \det S(s_2) \leq c_1^{2n_1}c_2^{2n_2} \det S(s^h) \det S(s^h) \leq c_1^{2n_1}(1 - c_1^2)^{n_1} \det S(s^h) \det S(s^h) \leq \left( \frac{n_1}{n_1 + n_2} \right)^{n_1} \left( \frac{n_2}{n_1 + n_2} \right)^{n_2} \det S(s^h) \det S(s^h) = \det S(s_\ast),
\]
since the maximum of the function \( f(t) = t^{n_1}(1 - t)^{n_2}, \ 0 \leq t \leq 1, \) is attained at \( t = n_1/(n_1 + n_2). \)

For a domain \( G \subset \mathbb{C}^{n} \) and \( \eta \in \mathcal{M}(G, \mathbb{C}) \) (cf. § 1.2.1), we define the \textit{Wu pseudometric}
\[
(\mathbb{W}\eta)(a; X) := (\mathbb{W}\tilde{\eta}(a; \cdot))(X), \quad (a, X) \in G \times \mathbb{C}^{n},
\]
where \( \tilde{\eta} \) is the Buseman pseudometric associated to \( \eta \) (cf. § 1.2.3). Observe that \( \mathbb{W}\eta \in \mathcal{M}(G, \mathbb{C}). \)

Recall that a Borel measurable metric \( \eta \in \mathcal{M}(G, \mathbb{C}) \) is said to be \textit{complete} if any \( \mathbb{W}\eta \)–Cauchy sequence is convergent to a point from \( G \) (cf. [J-P 1993], § 7.3).
Proposition 1.2.13. (a) If $\eta \in \mathcal{M}(G, \mathbb{C})$ is a continuous metric, then so is $W\eta$ (cf. Example 1.2.15).
(b) If $\eta \in \mathcal{M}(G, \mathbb{C})$ is a continuous complete metric, then so is $W\eta$.
(c) If $(\delta_G)_G$ is a holomorphically contractible family of pseudometrics, then:
- for any biholomorphic mapping $F : G \to D, G, D \subset \mathbb{C}^n$, we have
  $$(W\delta_D)(F(z); F'(z)(X)) = (W\delta_G)(z; X), \quad (z, X) \in G \times \mathbb{C}^n;$$
- for any holomorphic mapping $F : G \to D, G \subset \mathbb{C}^n, D \subset \mathbb{C}^{n_2}$, we have
  $$(\mathcal{W}\delta_D)(F(z); F'(z)(X)) \leq \sqrt{m}(W\delta_G)(z; X), \quad (z, X) \in G \times \mathbb{C}^{n_1},$$

but, for example, the family $(W\mathcal{X})_G$ is not holomorphically contractible (cf. Example 1.2.14).

In the case $n = \mathcal{X}_G$, the above properties (a) — (c) were formulated (without proof) in [Wu], [Wu 1993].

Proof. (a) Fix a point $z_0 \in G \subset \mathbb{C}^n$. Let $s_z := s^\mathcal{H}(z; -), z \in G$. We are going to show that $s_z \to s_{z_0}$ when $z \to z_0$.

By our assumptions, there exist $r > 0, c > 0$ such that
$$\eta(z; X) \geq c\|X\|, \quad z \in B(z_0, r) \subset G, X \subset \mathbb{C}^n.$$

In particular, the sets
$$I_z := \{ X \in \mathbb{C}^n : \tilde{\eta}(z; X) < 1 \}, \quad z \in B(z_0, r),$$
are contained in the ball $B(C)$ with $C := 1/c$. Moreover,
$$|\tilde{\eta}(z; X) - \tilde{\eta}(z_0; X)| \leq \varphi(z)\|X\|, \quad X \subset \mathbb{C}^n,$$
where $\varphi(z) \to 0$ when $z \to z_0$. Hence
$$(1 + C\varphi(z))^{-1}I_z \subset I_{z_0} \subset (1 + C\varphi(z))I_z, \quad z \in B(z_0, r),$$
and, consequently,
$$I_{z_0} \subset (1 + C\varphi(z))B(s_z) = B((1 + C\varphi(z))^{-2}s_z), \quad (1.2.5)$$
$$I_z \subset (1 + C\varphi(z))B(s_{z_0}) = B((1 + C\varphi(z))^{-2}s_{z_0}), \quad z \in B(z_0, r).$$

Hence,
$$\Vol(s_{z_0}) \leq \Vol((1 + C\varphi(z))^{-2}s_z) = (1 + C\varphi(z))^{2n}\Vol(s_z),$$
$$\Vol(s_z) \leq \Vol((1 + C\varphi(z))^{-2}s_{z_0}) = (1 + C\varphi(z))^{2n}\Vol(s_{z_0}), \quad z \in B(z_0, r).$$
Thus $\Vol(s_z) \to \Vol(s_{z_0})$ when $z \to z_0$.

Take a sequence $z_n \to z_0$. Since
$$|s_{z_n}(e_j, e_k)| \leq \eta(z_{\nu}; e_j)\eta(z_{\nu}; e_k), \quad j, k = 1, \ldots, n, \quad \nu \in \mathbb{N},$$
we may assume that $s_{z_n} \to s_\ast$, where $s_\ast$ is a pseudo–Hermitian scalar product. We already know that $\Vol(s_\ast) = \Vol(s_{z_0})$. Moreover, by (1.2.5), $I_{z_0} \subset B(s_\ast)$. Consequently, the uniqueness of $s_{z_0}$ implies that $s_\ast = s_{z_0}$.
(b) Recall that \( \int \eta = \int \hat{\eta} - \text{cf. [J-P 1993], Proposition 4.3.5(b).} \) By (a), \( \mathbb{W}_\eta \) is a continuous metric. In particular, the distance \( \int(\mathbb{W}_\eta) \) is well defined. By Proposition 1.2.12(a) we get

\[
\int \hat{\eta} \leq \int(\mathbb{W}_\eta),
\]

which directly implies the required result.

(c) Recall that the family \( (\delta_G)_G \) is holomorphically contractible (cf. §1.2.3).

In the general case, using Proposition 1.2.12(a), we get

\[
(\mathbb{W}(\delta_D))(F(z); F'(z)(X)) \leq \sqrt{n_2} \delta_D(F(z); F'(z)(X)) \leq \sqrt{n_2}(\mathbb{W}(\delta_D))(z; X), \quad (z, X) \in G \times \mathbb{C}^n. \]

\[\square\]

Example 1.2.14. Let \( G_\varepsilon := \{ (z_1, z_2) \in \mathbb{B}_2 : |z_1| < \varepsilon \}, 0 < \varepsilon < 1/\sqrt{2}. \) Recall that \( \mathbb{K}_{B_2}(0; X) = \|X\| \) and \( \mathbb{K}_{G_0}(0; X) = \max\{\|X\|, |X_1|/\varepsilon\}, X = (X_1, X_2). \) Then

\[
(\mathbb{W}(\mathbb{K}_{G_0}))(0; (X_1, X_2)) = \sqrt{|X_1|^2/\varepsilon^2 + |X_2|^2/(1 - \varepsilon^2)}, \quad X = (X_1, X_2) \in \mathbb{C}^2.
\]

In particular,

\[
(\mathbb{W}(\mathbb{K}_{B_2}))(0; (0, 1)) = \sqrt{2} > \frac{1}{\sqrt{1 - \varepsilon^2}} = (\mathbb{W}(\mathbb{K}_{G_0}))(0; (0, 1)).
\]

Consequently, the family \( (\mathbb{W}(\mathbb{K}_{G}))_G \) is not contractible with respect to inclusions.

We point out that Proposition 1.2.13(a) gives us the continuity of \( \mathbb{W}_\eta \) only in the case where \( \eta \) is a continuous metric. The following Example 1.2.15 shows that if \( \eta \) is only upper semicontinuous, then \( \mathbb{W}_\eta \) need not be upper semicontinuous. We do not know whether \( \mathbb{W}_\eta \) is upper semicontinuous in the case where \( \eta \) is a continuous pseudometric. Observe that the upper semicontinuity (or at least Borel measurability) of \( \mathbb{W}_\eta \) appears in a natural way when one defines \( \int(\mathbb{W}_\eta) \). In the case where \( \eta = \mathbb{K}_G \), the upper semicontinuity of \( \mathbb{W}(\mathbb{K}_G) \) is claimed for instance in [Wu 1993] (Theorem 1), [Che-Kim 1996] (Proposition 2), [Juc 2002] (Theorem 0), but so far there is no proof.

[?] Let \( \eta \in \{ \gamma^{(b)}_G, A_G, \mathbb{K}_G \} \) (cf. §1.2). Is \( \mathbb{W}_\eta \) upper semicontinuous [?]

Example 1.2.15. There is an upper semicontinuous metric \( \eta \) such that \( \mathbb{W}_\eta \) is not upper semicontinuous.

Indeed, let \( \eta : \mathbb{B}_2 \times \mathbb{C}^2 \longrightarrow \mathbb{R}_+ \), \( \eta(z; X) := \|X\| \) for \( z \neq 0 \), and \( \eta(0; X) := \max\{\|X\|, |X_1|/\varepsilon\}, X = (X_1, X_2) \in \mathbb{C}^2 (\varepsilon > 0 \text{ small}). \) Then \( (\mathbb{W}_\eta)(z; X) = \sqrt{2}\|X\| \) for \( z \neq 0 \), and (by Example 1.2.14) \( \{ X \in \mathbb{C}^2 : (\mathbb{W}_\eta)(0; X) < 1 \} \not\subset \mathbb{B}(1/\sqrt{2}) \), so \( \mathbb{W}_\eta \) is not upper semicontinuous.

Example 1.2.16. There exists a bounded domain \( G \subset \mathbb{C}^2 \) such that \( \mathbb{W}(\mathbb{K}_G) \) is not continuous (see Proposition 2 in [Che-Kim 1996], where such a continuity is claimed).

Indeed, let \( D \subset \mathbb{C}^2 \) be a domain such that (cf. [J-P 1993], Example 3.5.10):

- there exists a dense subset \( M \subset \mathbb{C} \) such that \( (M \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}) \subset D \),
- \( \mathbb{K}_D(z; (0, 1)) = 0, z \in A := M \times \mathbb{C} \),
1.2. Holomorphically contractible families of pseudometrics

• there exists a point \( z^0 \in D \setminus A \) such that \( \varphi_D(z^0; X) \geq c\|X\| \), \( X \in \mathbb{C}^2 \), where \( c > 0 \) is a constant.

For \( R > 0 \) let \( D_R := \{ z = (z_1, z_2) \in D : |z_j - z^0_j| < R, \ j = 1, 2 \} \). It is known that \( \varphi_{D_R} \setminus \varphi_D \) when \( R \not\rightarrow +\infty \). Observe that \( z^0 \in D_R \) and

\[
\varphi_{D_R}(z^0; X) \geq \varphi_D(z^0; X) \geq c\|X\|, \quad X \in \mathbb{C}^2.
\]

Hence, by Proposition 1.2.12(a), \( (\mathbb{W}_{\varphi_{D_R}})(z^0; X) \geq \|X\|, \ X \in \mathbb{C}^2 \). In particular,

\[
(\mathbb{W}_{\varphi_{D_R}})(z^0; (0, 1)) \geq c.
\]

Fix a sequence \( M \ni z_k \rightarrow z^0 \). Note that \( \{ z_k \} \times (z^0_2 + RE) \subset D_R \), which implies that \( \varphi_{D_R}((z_k, z^0_2); (0, 1)) \leq 1/R, \ k = 1, 2, \ldots \). In particular,

\[
(\mathbb{W}_{\varphi_{D_R}})((z_k, z^0_2); (0, 1)) \leq \sqrt{2} \varphi_{D_R}((z_k, z^0_2); (0, 1)) \leq \sqrt{2}/R, \quad k = 1, 2, \ldots
\]

Now it clear that if \( R > \frac{\sqrt{2}}{c} \), then

\[
\limsup_{k \rightarrow +\infty}(\mathbb{W}_{\varphi_{D_R}})((z_k, z^0_2); (0, 1)) \leq \frac{\sqrt{2}}{R} < c \leq (\mathbb{W}_{\varphi_{D_R}})(z^0; (0, 1)),
\]

which shows that for \( G := D_R \) the pseudometric \( \mathbb{W}_{\varphi_G} \) is not continuous.

**Remark 1.2.17.** We point out the role played in the definition of \( \mathbb{W} \) by the factor \( \sqrt{m} \).

Put \( \mathbb{W}_{\varphi, m} := \varphi_{g_{\varphi, m}}, \mathbb{W}_{\varphi, m}(a; X) = (\mathbb{W}_{\varphi, m}(a; \cdot))(X), (a, X) \in G \times \mathbb{C}^n \). Let \( D \subset \mathbb{C}^2 \) and \( D \ni z_k \rightarrow z_0 \in D \) be such that:

• \( \varphi_{D}(z_k; \cdot) \) is not a metric (in particular, \( m(k) := \dim U(z_k; \cdot) \leq 1, \ k \in \mathbb{N} \)),

• \( \varphi_{D}(z_0; \cdot) \) is a metric (take, for instance, the domain \( D \) from Example 1.2.16).

Put \( G := D \times E \subset \mathbb{C}^{3} \). Then

\[
(\mathbb{W}_{\varphi, m})^2((z_k, 0); (0, 1)) = s_{\varphi_{G}}((z_k, 0); (0, 1), (0, 1)) = \frac{1}{m(z_k) + 1} \geq \frac{1}{2}, \ k \in \mathbb{N},
\]

\[
(\mathbb{W}_{\varphi, m})^2((z_0, 0); (0, 1)) = s_{\varphi_{G}}((z_0, 0); (0, 1), (0, 1)) = \frac{1}{m(z_0) + 1} = \frac{1}{3},
\]

and, therefore, \( \mathbb{W}_{\varphi, m} \) is not upper semicontinuous at \( (z_0, 0), (0, 1) \) (the example is due to W. Jarnicki).

**Remark 1.2.18.** The Wu metric in complex ellipsoids \( \mathbb{E}_{(1, m)} \) was studied in [Che-Kim 1996] \( (m \geq \frac{1}{2}) \) and [Che-Kim 1997] \( (0 < m < \frac{1}{2}) \).

In a recent paper [Che-Kim 2003] the same authors proved the following two results.

Let \( G := \mathbb{C}^n \setminus U \), where \( U \) is open in \( \mathbb{C}^n \). Then there exists a neighborhood \( V \) of \( \partial G \cap \partial \mathbb{B}^n \) such that \( \mathbb{W}_{\varphi_G} = \mathbb{W}_{\varphi_{\mathbb{E}^n}} \) in \( V \cap G \).

Let \( p = (p_1, \ldots, p_n) \in \mathbb{N}^n, p_j \geq 2, j = 1, \ldots, n \). Then any strongly pseudoconvex point \( a \in \partial \mathbb{E}_p \) has a neighborhood \( V \) such that \( \mathbb{W}_{\varphi_p} \) is a Kähler metric with constant negative curvature in \( V \cap \mathbb{E}_p \).

1.2.7. Regularity of contractible pseudodistances and pseudometrics. Let us mention a few new results related to different regularity properties of contractible objects.

• Let \( (G_j)_{j=1}^\infty \) be a sequence of domains in \( \mathbb{C}^n \) such that \( G_{j+1} \subset G_j \) and \( \bigcap_{j=1}^\infty G_j = G \), where \( G \) is a domain in \( \mathbb{C}^n \). It is an open question to find conditions under which
M. Kobayashi in [KobM 2002] proved the following two results:

(a) If $G$ is strongly pseudoconvex, then $c_G \rightarrow c_G$ locally uniformly.
(b) If $G$ is a bounded domain such that every point $b \in \partial G$ admits a weak peak function (i.e., a function $f$ holomorphic in a neighborhood of $G$ such that $f(b) = 1$ and $|f| < 1$ on $G$), then $k_G \rightarrow k_G$ locally uniformly.

- The behavior of the Bergman, Carathéodory, and Kobayashi metrics on a smooth bounded pseudoconvex domain $G \subset \mathbb{C}^n$ near a boundary point of finite type, where the Levi form of $\partial G$ has at least $n - 2$ positive eigenvalues, was studied in [Cho 1995].

- Lower and upper nontangential bounds for the Carathéodory metric of a smooth bounded pseudoconvex domain $G \subset \mathbb{C}^n$ near an $h$–extendible boundary point (a boundary point is said to be $h$–extendible if its Catlin multitype coincides with its D’Angelo type) were proved in [Nik 1997] and [Nik 1999].

- Some localization theorems for contractible functions and metrics were proved in [Nik 2002].

- Let $G$ be a strongly pseudoconvex balanced domain with $C^\infty$ (resp. real analytic) boundary. Then there is an open neighborhood $U = U(0) \subset G$ such that $\kappa_G$ is $C^\infty$ (resp. real analytic) on $U \times (\mathbb{C}^n)_*$; cf. [Pan 1993].

- Let $D_m := \mathbb{E}_{(1,m)} \times (\mathbb{C}^2)_* \subset \mathbb{C}^4$, $m > 0$. It was proved in [Ma 1995] that:
  (a) $\kappa_{\mathbb{E}_{(1,m)}} \in C^2(D_m)$ for $m \geq 1$,
  (b) $\kappa_{\mathbb{E}_{(1,m)}}$ is piecewise $C^3$ on $D_m$ and $\kappa_{\mathbb{E}_{(1,m)}} \notin C^3(D_m)$ for $m \geq \frac{3}{2}$.

- Let $G, D \subset \mathbb{C}^n$ be domains and let $a \in G$, $b \in D$. We say that a holomorphic mapping $F : G \rightarrow D$ with $F(a) = b$ is Carathéodory extremal if
  \[ |\det F'(a)| = \sup\{|\det \Phi'(a)| : \Phi \in \mathcal{O}(G, D), \Phi(a) = b\}. \]

In the cases:
\begin{align*}
G &= \mathbb{B}_n, D = \mathbb{E}_{m,p}, a = b = 0, \\
G &= \mathbb{E}_{m,p}, D = \mathbb{B}_n, a = b = 0,
\end{align*}
where
\[ \mathbb{E}_{m,p} := \left\{ z \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k} : \sum_{j=1}^k ||z_j||^{2p} < 1 \right\}, \]
\[ m = (m_1, \ldots, m_k) \in \mathbb{N}^k, m_1 + \cdots + m_k = n, p = (p_1, \ldots, p_k) \in \mathbb{R}_{>0}^n, \]
the Carathéodory extremal mappings are characterized in [Ma 1997].
1.3. Effective formulas for elementary Reinhardt domains

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ and $c \in \mathbb{R}$ put

$$D_{\alpha,c} := \{ z \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < e^c (\forall j \in \{1, \ldots, n\} : \alpha_j < 0 \implies z_j \neq 0) \}, \quad D_{\alpha} := D_{\alpha,0};$$

$D_{\alpha,c}$ is called an elementary Reinhardt domain. We say that $D_{\alpha,c}$ is of rational type if $\alpha \in \mathbb{R} \cdot \mathbb{Z}^n$. The domain $D_{\alpha,c}$ is of irrational type if it is not of rational type. Without loss of generality we may assume that $\alpha_1, \ldots, \alpha_k < 0$ and $\alpha_{k+1}, \ldots, \alpha_n > 0$ for some $k \in \{0, \ldots, n\}$. If $k < n$, then we put $t_k := \min\{\alpha_{k+1}, \ldots, \alpha_n\}$. Let

$$V_0 := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0 \}.$$

For $\alpha \in \mathbb{Z}^n$ and $r \in \mathbb{N}$, put $\Phi(z) := z^\alpha$,

$$\Phi_{(r)}(a)(X) := \sum_{\beta \in \mathbb{Z}_{+}^n, |\beta| = r} \frac{1}{\beta!} D^\beta \Phi(a) X^\beta, \quad a \in D_{\alpha}, \ X \in \mathbb{C}^n.$$ 

To simplify notation, for $z \in D_{\alpha}$, write $|z^\alpha| := |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}$ (observe that this notation agrees with the standard one if $\alpha \in \mathbb{Z}^n$).

The following effective formulas for holomorphically contractible functions and pseudometrics on $D_{\alpha}$ are known.

**Theorem 1.3.1** ([J-P 1993] (§ 4.4), [Pfl-Zwo 1998], [Zwo 1999a], [Zwo 2000a]). Let $a = (a_1, \ldots, a_n) \in D_{\alpha}$. Assume that $a_1 \cdots a_s \neq 0$, $a_{s+1} = \cdots = a_n = 0$ for some $s \in \{k+1, \ldots, n\}$. Put $r := \ord_{\alpha}(z^\alpha - a^\alpha)$. For $z \in D_{\alpha}$ and $X \in \mathbb{C}^n$ consider the following four cases.

1. $k < n$, $D_{\alpha}$ is of rational type (we may assume that $\alpha \in \mathbb{Z}^n$ and $\alpha_1, \ldots, \alpha_n$ are relatively prime). Then:

   $$c'_{D_{\alpha}}(a, z) = m_E(a^\alpha, z^\alpha),$$
   $$g'_{D_{\alpha}}(a, z) = (m_E(a^\alpha, z^\alpha))^{1/r},$$
   $$k_{D_{\alpha}}(a, z) = \min\{ pE(\zeta_1, \zeta_2) : \zeta_1, \zeta_2 \in E, \ a^\alpha = \zeta_1^{t_1}, z^\alpha = \zeta_2^{t_2} \},$$
   $$\tilde{k}_{D_{\alpha}}(a, z) = \begin{cases} \min\{ pE(\zeta_1, \zeta_2) : \zeta_1, \zeta_2 \in E, \ a^\alpha = \zeta_1^{t_1}, z^\alpha = \zeta_2^{t_2} \}, & s = n, \ z \notin V_0 \\ pE(0, |z^\alpha|^{1/r}), & s < n \end{cases},$$

   $$\gamma_{D_{\alpha}}(a; X) = \gamma_E \left( a^\alpha, a^\alpha \sum_{j=1}^{n} \frac{\alpha_j X_j}{a_j} \right),$$
   $$A_{D_{\alpha}}(a; X) = (\gamma_E(a^\alpha, \Phi_{(r)}(a)(X)))^{1/r},$$
   $$\kappa_{D_{\alpha}}(a; X) = \begin{cases} \gamma_E \left( (a_1)^{1/t_1}; (a_2)^{1/t_2} \sum_{j=1}^{n} \frac{\alpha_j X_j}{a_j} \right), & s = n \\ \left( |a_1|^{\alpha_1} \cdots |a_s|^{\alpha_s} |X_{s+1}|^{\alpha_{s+1}} \cdots |X_n|^{\alpha_n} \right)^{1/r}, & s < n \end{cases}. $$
(2) $k < n$, $D_\alpha$ is of irrational type (we may assume that $t_k = 1$). Then:

\[ c^*_D = m^{(t)}_{D_\alpha} = 0, \quad \ell \in \mathbb{N}, \]
\[ g_{D_\alpha}(a, z) = \begin{cases} 0, & s = n \\ |z^\alpha|^{1/r}, & s < n \end{cases}, \]
\[ k_{D_\alpha}(a, z) = p_E(|a^\alpha|, |z^\alpha|), \]
\[ \tilde{k}_{D_\alpha}(a, z) = \begin{cases} p_E(|a^\alpha|, |z^\alpha|), & s = n, z \notin V_0 \\ p_E(0, |z^\alpha|^{1/r}), & s < n \end{cases}, \]
\[ \gamma_{D_\alpha} = \gamma^{(t)}_{D_\alpha} = 0, \quad \ell \in \mathbb{N}, \]
\[ A_{D_\alpha}(a; X) = \begin{cases} 0, & s = n \\ ([a_1]^{\alpha_1} \cdots [a_n]^{\alpha_n} |X_{s+1}|^{\alpha_{s+1}} \cdots |X_n|^{\alpha_n})^{1/r}, & s < n \end{cases}, \]
\[ \kappa_{D_\alpha}(a; X) = \gamma_E\left(\begin{bmatrix} a_1^{\alpha_1} \\ \vdots \\ a_n^{\alpha_n} \end{bmatrix} \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right), \quad s = n \]
\[ \kappa_{D_\alpha}(a; X) = \gamma_E\left(\begin{bmatrix} a_1^{\alpha_1} \\ \vdots \\ a_n^{\alpha_n} \end{bmatrix} \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right), \quad s < n. \]

(3) $k = n$, $D_\alpha$ is of rational type (we may assume that $\alpha \in \mathbb{Z}_n$ and $\alpha_1, \ldots, \alpha_n$ are relatively prime). Then:

\[ c^*_{D_\alpha}(a, z) = g_{D_\alpha}(a, z) = m_E(a^\alpha, z^\alpha), \]
\[ k_{D_\alpha}(a, z) = \tilde{k}_{D_\alpha}(a, z) = k_E(a^\alpha, z^\alpha), \]
\[ \gamma_{D_\alpha}(a; X) = A_{D_\alpha}(a; X) = \gamma_E\left(\begin{bmatrix} a_1^{\alpha_1} \\ \vdots \\ a_n^{\alpha_n} \end{bmatrix} \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right), \]
\[ \kappa_{D_\alpha}(a; X) = \kappa_E\left(\begin{bmatrix} a_1^{\alpha_1} \\ \vdots \\ a_n^{\alpha_n} \end{bmatrix} \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right). \]

(4) $k = n$, $D_\alpha$ is of irrational type. Then:

\[ c^*_{D_\alpha} \equiv m^{(t)}_{D_\alpha} \equiv g_{D_\alpha} \equiv 0, \quad \ell \in \mathbb{N}, \]
\[ k_{D_\alpha}(a, z) = \tilde{k}_{D_\alpha}(a, z) = k_E(|a^\alpha|, |z^\alpha|), \]
\[ \gamma_{D_\alpha} \equiv \gamma^{(t)}_{D_\alpha} \equiv A_{D_\alpha} \equiv 0, \quad \ell \in \mathbb{N}, \]
\[ \kappa_{D_\alpha}(a; X) = \kappa_E\left(\begin{bmatrix} a_1^{\alpha_1} \\ \vdots \\ a_n^{\alpha_n} \end{bmatrix} \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right). \]
Moreover, if $\alpha \in \mathbb{N}^n$ and $\alpha_1, \ldots, \alpha_n$ are relatively prime, then
\[
m^{(\ell)}_{D\alpha}(a, z) = (m_E(a^{\alpha}, z^{\alpha}))^{1/\ell}, \quad (13)
\]
\[
\gamma^{(\ell)}_{D\alpha}(a; X) = \begin{cases} \langle \gamma_E(a^{\alpha}, \Phi_r(a)(X)) \rangle^{1/r} & \text{if } r \text{ divides } \ell, \\ 0, & \text{otherwise} \end{cases}, \quad \ell \in \mathbb{N}.
\]

**Remark 1.3.2.** The formulas in Theorem 1.3.1 led W. Zwonek [Zwo 1998] to a negative answer for a question posed by S. Kobayashi (cf. Remark 3.3.8(b) in [J-P 1993]). Let $(\alpha_1, \ldots, \alpha_{n-1}, -1) \in \mathbb{R}^n$, $\alpha_j < 0$, such that $D_\alpha$ is of irrational type. Then the following mapping
\[
\psi : \mathbb{C}^{n-1} \times E_s \longrightarrow D_\alpha, \quad \psi(z_1, \ldots, z_n) := \left( e^{-z_1}, \ldots, e^{-z_{n-1}}, \frac{e^{-(\alpha_1 z_1 + \cdots + \alpha_{n-1} z_{n-1})}}{z_n} \right),
\]
is a holomorphic covering. Fix points $z \in D_\alpha$ and $w := (z_1, \ldots, z_{n-1}, |z_n|) \in D_\alpha$. Then, in virtue of the formula of S. Kobayashi (see Theorem 3.3.7 in [J-P 1993]), the product property for the Kobayashi pseudodistance, and Theorem 1.3.1, we have
\[
0 = k_{D\alpha}(w, z) = \inf_{\ell_1, \ldots, \ell_{n-1} \in \mathbb{Z}} \left\{ k_E \left( \frac{|z_1|^{\ell_1} \cdots |z_{n-1}|^{\ell_{n-1}}}{|z_n|} e^{i \sum_{j=1}^{n-1} \arg(z_j) \alpha_j}, \frac{|z_1|^{\ell_1} \cdots |z_{n-1}|^{\ell_{n-1}}}{z_n} e^{i \sum_{j=1}^{n-1} (\arg(z_j) + 2\ell_j \pi) \alpha_j} \right) \right\}.
\]
Assuming that the infimum is attained implies that there are $\ell_1, \ldots, \ell_{n-1} \in \mathbb{Z}$ such that
\[
\frac{\arg(z_n)}{2\pi} + \sum_{j=1}^{n-1} \ell_j \alpha_j \in \mathbb{Z}. \quad \text{This is, in general, impossible. Just take a } z_n \text{ in such a way that } \frac{\arg(z_n)}{2\pi} \text{ does not belong to the } \mathbb{Q}-\text{linear subspace of } \mathbb{R} \text{ which is spanned by 1, } \alpha_1, \ldots, \alpha_{n-1}. \quad \text{Recall that } \mathbb{R} \text{ as a } \mathbb{Q}-\text{vector space is infinite dimensional.}
\]

**Remark 1.3.3.** The Wu pseudometric for $D_\alpha$ (with $\alpha \in \mathbb{R}_+^n$) was investigated by P. Jucha in [Juc 2002]. He proved that
\[
(\forall x_{D\alpha})(a; X) = \begin{cases} \sqrt{n} k_{D\alpha}(a; X) & \text{if } \# J(a) \leq 1, \\ 0 & \text{if } \# J(a) \geq 2, \quad (a, X) \in D_\alpha \times \mathbb{C}^n,
\end{cases}
\]
where $J(a) := \{ j \in \{1, \ldots, n\} : a_j = 0 \}, a \in D_\alpha$.

**Remark 1.3.4.** (a) Observe that if $\alpha \in \mathbb{N}^n$, $\alpha_1, \ldots, \alpha_n$ are relatively prime, and $t_0 = 1$, then
\[
c^{(t)}_{D\alpha} \equiv \tanh k^{(t)}_{D\alpha} \leq g_{D\alpha}, \quad \tanh k^{(t)}_{D\alpha} \neq g_{D\alpha}, \quad \gamma_{D\alpha} \leq k_{D\alpha}, \quad \gamma_{D\alpha} \neq k_{D\alpha}.
\]
(b) For a domain $G \subset \mathbb{C}^n$ define the following relation $R$:
\[
a R b : \iff k_G(a, b) = 0, \quad a, b \in G.
\]
In [Kob 1976], S. Kobayashi asked whether the quotient $G/R$ has always a complex structure. From Theorem 1.3.1 we see that if $D_\alpha$ is of irrational type with $k = 0$, then $D_\alpha/R \approx [0, 1]$. This gives a simple example of a very regular domain for which the answer to the above question is “No”.

\[\text{Remark 1.3.5.} \quad [x] := \inf \{ m \in \mathbb{Z} : m \geq x \}.\]
1. Holomorphically invariant objects

1.4. The converse to the Lempert theorem

First recall the fundamental Lempert theorem saying that if a domain \( G \subset \mathbb{C}^n \) is strongly linearly convex, then \( c_G^* \equiv \tilde{k}_G^* \) (cf. [J-P 1993], Miscellanea C; recall that any strongly convex domain is strongly linearly convex). Notice that if for a domain \( G \subset \mathbb{C}^n \) we have \( c_G^* \equiv \tilde{k}_G^* \), then, by (1.1.2), all holomorphically contractible families coincide on \( G \). Moreover, if \( G \) is taut and \( c_G^* \equiv \tilde{k}_G^* \), then \( \gamma_G \equiv \kappa_G \) (cf. Proposition 1.2.6) and, consequently, all holomorphically contractible families of pseudometrics coincide on \( G \).

Note that in the case of convex domains \( G \subset \mathbb{C}^n \), the equality \( c_G^* \equiv \tilde{k}_G^* \) may be also proved using functional analysis methods; cf. [Mey 1997].

Let \( L_n \) be the class of all domains \( G \subset \mathbb{C}^n \) with \( c_G^* \equiv \tilde{k}_G^* \). It is clear that \( L_n \) is invariant under biholomorphic mappings. Moreover, if a domain \( G \subset \mathbb{C}^n \) may be exhausted by domains from \( L_n \) (i.e. \( G = \bigcup_{i \in I} G_i, G_i \in L_n, i \in I \), and for any compact \( K \subset G \) there exists an \( i_0 \in I \) with \( K \subset G_{i_0} \)), then \( G \in L_n \).

Indeed, we only need to prove that
\[
d_G = \inf \{ d_{G_i} : i \in I \}, \quad d \in \{ c^*, \tilde{k}^* \}.\]

Write \( G = \bigcup_{k=1}^{\infty} G'_k \), where \( G'_k \subset G \) is a domain with \( G'_k \subset G'_{k+1}, k \in \mathbb{N} \). For \( k \in \mathbb{N} \) let \( i(k) \in I \) be such that \( G'_k \subset G_{i(k)} \). Then \( d_{G'_k} \searrow d_G \) (cf. [J-P 1993], Propositions 2.5.1(a), 3.3.5(a)). Hence
\[
d_G \leq \inf \{ d_{G_i} : i \in I \} \leq \inf \{ d_{G_{i(k)}} : k \in \mathbb{N} \} \leq \inf \{ d_{G'_k} : k \in \mathbb{N} \} = d_G.
\]

For example, if \( G = \bigcup_{k=1}^{\infty} G_k \) with \( G_k \subset G_{k+1}, G_k \in L_n, k \in \mathbb{N} \), then \( G \in L_n \).

In particular, any convex domain belongs to \( L_n \).

For more than 20 years the following conjecture was open. Any bounded pseudoconvex domain \( G \in L_n \) may be exhausted by domains biholomorphic to convex domains (\(^{14}\))

For instance, it is unknown whether the strongly linearly convex domain
\[
G = \{ z \in \mathbb{C}^n : \|z\|^2 + (\text{Re}(z_1))^2 < 1 \}
\]

may be exhausted by domains biholomorphic to convex domains; even more, it is not known whether \( G \) is biholomorphic to a convex domain (cf. [J-P 1993], Example C.3).

The first counterexample was recently constructed in a series of papers by J. Agler, C. Costara, and N. J. Young [Agl-You 2001], [Agl-You 2003], [Cos 2003a], [Cos 2003b], [Cos 2004], [Ham-Seg 2003] when they investigated the \( 2 \times 2 \)-spectral Nevanlinna–Pick

\(^{14}\) Observe that there are unbounded pseudoconvex domains \( G \in L_n \) which cannot be exhausted by convex domains, e.g. \( G = \mathbb{C}_* \) (\( c_{\mathbb{C}_*}^* \equiv \tilde{k}_{\mathbb{C}_*}^* \equiv 0 \)).
problem. Let
\[ \pi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \quad \pi(\lambda_1, \lambda_2) := (\lambda_1 + \lambda_2, \lambda_1 \lambda_2), \]
\[ G_2 := \pi(E^2) = \{(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) : \lambda_1, \lambda_2 \in E\}, \quad \sigma_2 := \pi((\partial E)^2) \subset \partial G_2, \]
\[ \Delta_2 := \{(\lambda, \lambda) : \lambda \in E\}, \quad \Sigma_2 := \pi(\Delta_2) = \{(2\lambda, \lambda^2) : \lambda \in E\}, \]
\[ h_a(\lambda) := \frac{\lambda - a}{1 - \overline{a}\lambda}, \quad a \in E, \lambda \in \mathbb{C} \setminus \{1/\pi\}, \]
\[ F_a(s, p) := \frac{2ap - s}{2 - as}, \quad a \in E, (s, p) \in (\mathbb{C} \setminus \{2/a\}) \times \mathbb{C}. \]

Note that \( \pi \) is proper, \( \pi|_{E^2} : E^2 \longrightarrow G_2 \) is proper, and
\[ \pi|_{E^2 \setminus \Delta_2} : E^2 \setminus \Delta_2 \longrightarrow G_2 \setminus \Sigma_2 \]
is a holomorphic covering. The domain \( G_2 \) is called the symmetrized bidisc. One can prove (cf. Lemma 1.4.2) that \(|s| < 2\) and \(|F_a| < 1\) on \( G_2 \) and that (Remark 1.4.5) \( G_2 \) is hyperconvex.

**Theorem 1.4.1.** (15) We have
\[ c_{G_2}((s_1, p_1), (s_2, p_2)) = \tilde{c}_{G_2}((s_1, p_1), (s_2, p_2)) = \max\{m_E(F_z(s_1, p_1), F_z(s_2, p_2)) : z \in \overline{E}\} \]
\[ = \max\{m_E(F_z(s_1, p_1), F_z(s_2, p_2)) : z \in \partial E, (s_1, p_1), (s_2, p_2) \in G_2\}. \]
Moreover, \( G_2 \) cannot be exhausted by domains biholomorphic to convex domains.

The proof will be given after auxiliary lemmas.

**Lemma 1.4.2 ([Agl-You 2003]).** For \((s, p) \in \mathbb{C}^2\), the following conditions are equivalent:

(i) \((s, p) \in G_2\);
(ii) \(|s - \overline{p}| + |p|^2 < 1\);
(iii) \(|s| < 2\), \(|s - \overline{p}| + |p|^2 < 1\);
(iv) \(|\frac{2p^2 - s}{2 - \overline{p}z}| < 1\), \(z \in \overline{E}\) (i.e. \(|F_z(s, p)| < 1\), \(z \in \overline{E}\));
(v) \(|\frac{2p^2 - s}{2 - \overline{p}z}| < 1\), \(z \in \overline{E}\);
(vi) \(2|s - \overline{p}| + |s^2 - 4p| + |s|^2 < 4\).

In particular, \(F_a \in \mathcal{O}(G_2, E)\) \((a \in \overline{E})\).

**Proof.** Observe that \((s, p) \in G_2\) iff both roots of the polynomial \(f(z) = z^2 - sz + p \) belong to \(E\). By the Cohn criterion (cf. [Rah-Sch 2002]) \(f^{-1}(0) \subset E\) iff \(|p| < 1\) and the root of the polynomial
\[ g(z) := \frac{1}{2} \left( f(z) - \frac{1}{2} f(1/\overline{z}) \right) = (1 - |p|^2)z - (s - p\overline{s}) \]
belongs to \(E\). Thus (i) \(\iff\) (ii).

\(15\) Special thanks are due to C. Costara and N. J. Young for sending us their preprints without which this section would have been never written.
The equivalence (iv) ⇐⇒ (v) follows from the maximum principle:
\[
\max \left\{ \left| \frac{2zp - s}{2 - zs} \right| : z \in \overline{E} \right\} = \max \left\{ \left| \frac{2zp - s}{2 - zs} \right| : z \in \partial E \right\} \]
\[
= \max \left\{ \left| \frac{2p - zs}{2 - zs} \right| : z \in \partial E \right\} = \max \left\{ \left| \frac{2p - zs}{2} \right| : z \in \overline{E} \right\}.
\]

Observe that (iv) with \(z = 0\) gives \(|s| < 2\) and, moreover,
\[
\max \left\{ \left| \frac{2p - zs}{2 - zs} \right|^2 : z \in \partial E \right\} = \max \left\{ \frac{4|p|^2 + |s|^2 - 4 \text{Re}(z p \overline{s})}{4 + |s|^2 + 4 \text{Re}(z s)} : z \in \partial E \right\}.
\]

Thus (iv) \(\Rightarrow\) (iii) \(\Rightarrow\) (ii) \(\Rightarrow\) (iv).

Observe that for \(|s| < 2\) the mapping \(\overline{E} \ni z \mapsto F_z(s,p)\) maps \(\overline{E}\) onto \(B(a,r) \subset \mathbb{C}\) with \(a := \frac{2p - z s}{4 - |s|^2}, r := \frac{|s|^2 - 4|p|^2}{4 - |s|^2}\). We have \(B(a, r) \subset E\) if \(|a| + r < 1\). Thus (vi) \(\iff\) (iv).

\[\text{Lemma 1.4.3 ([Agl-You 2003])}.\]

For \((s,p) \in \mathbb{C}^2\), the following conditions are equivalent:

(i) \((s,p) \in \overline{G}_2\);
(ii) \(|s| \leq 2, |s - \overline{p}| + |p|^2 \leq 1;\)
(iii) \(|\frac{2s - p}{2 - zs}| \leq 1, z \in \overline{E}\) \((^{16})\);
(iv) \(|\frac{2p - zs}{2 - zs}| \leq 1, z \in \overline{E}\) \((^{17})\).

Notice that the condition \(|s - \overline{p}| + |p|^2 \leq 1\) does not imply that \((s,p) \in \overline{G}_2\) (e.g. \((s,p) = (5/2, 1))

\[\text{Proof.}\]

Using Lemma 1.4.2 we see that (i) \(\implies\) (ii). Moreover, (ii) \(\iff\) (iii) \(\implies\) (iv). It remains to observe that if (ii) is satisfied and \(s \neq \overline{p}\), then \((ts, s) \in \overline{G}_2, 0 < t < 1\). If (ii) is satisfied and \(s = \overline{p}\), then \((ts, t^2 p) \in \overline{G}_2, 0 < t < 1\).

\[\text{Corollary 1.4.4.}\]

For \((s,p) \in \mathbb{C}^2\), the following conditions are equivalent:

(i) \((s,p) \in \sigma_2;\)
(ii) \(s = \overline{p}, |p| = 1, \text{ and } |s| \leq 2.\)
In particular, \(|F_a| = 1\) on \(\sigma_2\) for \(a \in \partial E\) \((^{18})\).

\[\text{Remark 1.4.5.}\]

\(\rho(s,p) := \max \left\{ \max\{|\lambda_1|, |\lambda_2| : (\lambda_1, \lambda_2) \in \pi^{-1}(s,p)\} : (s,p) \in \mathbb{C}^2.\right\}\)

Then \(\rho\) is a continuous plurisubharmonic function such that
\(\rho(\lambda s, \lambda^2 p) = |\lambda|\rho(s,p), (s,p) \in \mathbb{C}^2, \lambda \in \mathbb{C},\)

and
\(G_2 = \{(s,p) \in \mathbb{C}^2 : \rho(s,p) < 1\},\)
\(\overline{G}_2 = \{(s,p) \in \mathbb{C}^2 : \rho(s,p) \leq 1\}.
\)

In particular, \(G_2\) is hyperconvex.

\[\text{Remark 1.4.5.}\]

Notice that if \((s,p) \in \overline{G}_2\) and \(|s| = 2\), then \((s,p) = (2\eta, \eta^2)\) for some \(\eta \in \partial E\) and, consequently, \(\frac{2s - p}{2 - zs} = -\eta\), which implies that the function \(z \mapsto \frac{2s - p}{2 - zs}\) has no essential singularities.

\[\text{Remark 1.4.5.}\]

As above, notice that if \((s,p) = (2\eta, \eta^2)\) for some \(\eta \in \partial E\), then \(\frac{2p - zs}{2 - zs} = \eta^2\) and, consequently, the function \(z \mapsto \frac{2p - zs}{2 - zs}\) has no essential singularities.

\[\text{Remark 1.4.5.}\]

Notice that for \((s,p) \in \sigma_2, |s| < 2, \text{ we have } F_a(s,p) = ap^2 - aps^{-1}.\]
1.4. The converse to the Lempert theorem

The maximum principle for plurisubharmonic functions gives the following result.

**Lemma 1.4.6.** Let \( \varphi : E \longrightarrow \mathbb{C}^2 \) be a continuous mapping, holomorphic in \( E \), such that \( \varphi(\partial E) \subset \mathcal{G}_2 \). Then \( \varphi(E) \subset \mathcal{G}_2 \). If \( \varphi(E) \cap \mathcal{G}_2 \neq \emptyset \), then \( \varphi(E) \subset \mathcal{G}_2 \). If \( \varphi(E) \cap \mathcal{G}_2 = \emptyset \), then \( \varphi(E) \subset \partial \mathcal{G}_2 \).

**Remark 1.4.7.** If \( f \in \mathcal{O}(E^2) \) is symmetric, then the relation \( F(\pi(\lambda_1, \lambda_2)) = f(\lambda_1, \lambda_2) \) defines a function \( F \in \mathcal{O}(\mathcal{G}_2) \).

In particular, if \( h \in \mathcal{O}(E, E) \), then the relation \( H_h(\pi(\lambda_1, \lambda_2)) = \pi(h(\lambda_1), h(\lambda_2)) \) defines a holomorphic mapping \( H_h : \mathcal{G}_2 \longrightarrow \mathcal{G}_2 \) with \( H_h(\Sigma_2) \subset \Sigma_2 \).

Observe that if \( h \in \text{Aut}(E) \), then \( H_h \in \text{Aut}(\mathcal{G}_2) \), \( H_h^{-1} = H_{h^{-1}} \), \( H_h(\Sigma_2) = \Sigma_2 \), and \( H_h(\sigma_2) = \sigma_2 \).

In particular, if \( h(\lambda) := \tau \lambda \) for some \( \tau \in \partial E \), then we get the “rotation” \( R_\tau(s, p) := H_h(s, p) = (\tau s , \tau^2 p) \).

**Remark 1.4.8.** For any point \((s_0, p_0) = (2a, a^2) \in \Sigma_2 \) we get \( H_{h_a}(s_0, p_0) = (0, 0) \).

**Lemma 1.4.9.** \( \sigma_2 \) is the Shilov (and Bergman) boundary of \( \mathcal{G}_2 \).

**Proof.** It is clear that the modulus of any function \( f \in \mathcal{O}(\mathcal{G}_2) \) attains its maximum on \( \sigma_2 \). We have to prove that \( \sigma_2 \) is minimal. First observe that the function \( f_0(s, p) := s + 2 \) is a peak function at \( (2, 1) \in \sigma_2 \). Take any other point \((s_0, p_0) = \pi(\lambda_1^0, \lambda_2^0) \in \sigma_2 \). The case \( \lambda_1^0 = \lambda_2^0 \) reduces (via a rotation \( R_\tau \)) to the previous one. Thus assume that \( \lambda_1^0 \neq \lambda_2^0 \). We are going to find a Blaschke product \( B \) of order \( 2 \) such that

\[
(\lambda \in \partial E : B(\lambda) = 1) = \{\lambda_1^0, \lambda_2^0\}.
\]

Suppose for a moment that such a \( B \) is already constructed. Then \( f_0 \circ H_B \) is a peak function for \((s_0, p_0) \).

We move to the construction of \( B \). Using a rotation we may reduce the proof to the case \( \lambda_2^0 = \overline{\lambda_1^0} \). Then, using the fact that the mapping \((-1, 1) \ni a \longrightarrow \frac{a + 1}{1 + a_2} \in (-1, 1) \) is bijective, we see that there exists an \( a \in (-1, 1) \) such that \( h_a(\lambda_1^0) = -h_a(\overline{\lambda_1^0}) \). Finally, we take \( B := \tau h_a^2 \) with \( \tau \in \partial E \) such that \( \tau h_a(\lambda_1^0) = 1 \). \( \square \)

**Lemma 1.4.10.** The domain \( \mathcal{G}_2 \) cannot be exhausted by domains biholomorphic to convex domains.

**Proof.** This is a generalization of the proof of [Cos 2003a] due to A. Edigarian [Edi 2003a].

First observe that \( \overline{\mathcal{G}_2} \) is not convex: for example, \((2, 1), (2i, -1) \in \overline{\mathcal{G}_2} \), but \((1 + i, 0) \notin \overline{\mathcal{G}_2} \). Consequently, \( \mathcal{G}_2 \) is also not convex.

Suppose that \( \mathcal{G}_2 = \bigcup_{i \in I} G_i \), where each domain \( G_i \) is biholomorphic to a convex domain and for any compact \( K \in \mathcal{G}_2 \) there exists an \( i_0 \in I \) with \( K \subset G_{i_0} \). For any \( 0 < \varepsilon < 1 \) take an \( i = i(\varepsilon) \in I \) such that \( \{ (s, p) \in \mathbb{C}^2 : \rho(s, p) \leq 1 - \varepsilon \} \subset G_{i(\varepsilon)} \) and let \( f_\varepsilon = (g_\varepsilon, h_\varepsilon) : G_{i(\varepsilon)} \longrightarrow D_\varepsilon \) be a biholomorphic mapping onto a convex domain \( D_\varepsilon \subset \mathbb{C}^n \) with \( f_\varepsilon(0, 0) = (0, 0) \) and \( f_\varepsilon'(0, 0) = iI_2 \).

Take arbitrary two points \((s_1, p_1), (s_2, p_2) \in \mathbb{C}^2 \) and put

\[
C := \max\{\rho(s_1, p_1), \rho(s_2, p_2)\}.
\]

Our aim is to prove that \( \rho(t(s_1, p_1) + (1 - t)(s_2, p_2)) \leq C, \ t \in [0, 1] \), which in particular shows that \( \mathcal{G}_2 \) is convex; contradiction.
Observe that for $|\lambda| < (1 - \varepsilon)/C$ we have $\rho(\lambda s_j, \lambda^2 p_j) = |\lambda|\rho(s_j, p_j) < 1 - \varepsilon$, $j = 1, 2$. Consequently, for any $t \in [0, 1]$, the mapping $\varphi_{t, t} : \mathbb{B}(\frac{1 - \varepsilon}{C}) \to \mathbb{G}_2$, 

$$
\varphi_{t, t}(\lambda) = (\psi_{t, t}(\lambda), \chi_{t, t}(\lambda)) := f_{\varepsilon}^{-1}(tf_{\varepsilon}(\lambda s_1, \lambda^2 p_1) + (1 - t)f_{\varepsilon}(\lambda s_2, \lambda^2 p_2))
$$

is well defined. We have $\varphi_{t, t}(0) = (0, 0)$ and, finally, 

$$
\frac{1}{2} \chi''_{t, t}(0) = tp_1 + (1 - t)p_2 + \mu_\varepsilon t(1 - t)(s_1 - s_2)^2,
$$

where $\mu_\varepsilon := \frac{1}{2} \frac{\partial^2 h_{\varepsilon}}{\partial s^2}(0, 0)$.

Define $\Phi_{t, t} : \mathbb{B}(\frac{1 - \varepsilon}{C}) \to \mathbb{C}^2$, 

$$
\Phi_{t, t}(\lambda) := \begin{cases} 
\left( \frac{1}{\lambda} \psi_{t, t}(\lambda), \frac{1}{\lambda} \chi_{t, t}(\lambda) \right), & \lambda \neq 0 \\
(t s_1 + (1 - t)s_2, tp_1 + (1 - t)p_2 + \mu_\varepsilon t(1 - t)(s_1 - s_2)^2), & \lambda = 0.
\end{cases}
$$

Then $\Phi_{t, t}$ is holomorphic and, by the maximum principle, we get 

$$
\rho(\Phi_{t, t}(0)) \leq \limsup_{s \to \frac{1 - \varepsilon}{C}} \max_{|\lambda| = s} \rho(\Phi_{t, t}(\lambda)) = \limsup_{s \to \frac{1 - \varepsilon}{C}} \max_{|\lambda| = s} (1 - |\chi_{t, t}(\lambda)|) \leq \frac{C}{1 - \varepsilon},
$$

i.e.

$$
\rho(t s_1 + (1 - t) s_2, t p_1 + (1 - t) p_2 + \mu_\varepsilon t(1 - t)(s_1 - s_2)^2) \leq \frac{C}{1 - \varepsilon}.
$$

We only need to prove that $\mu_\varepsilon \to 0$.

Taking $t = 1/2$ we get 

$$
\rho\left(\frac{1}{2}(s_1 + s_2), \frac{1}{2}(p_1 + p_2) + \frac{1}{4}\mu_\varepsilon(s_1 - s_2)^2\right) \leq \frac{C}{1 - \varepsilon}.
$$

For $\alpha \in \partial E$, take $(s_1, p_1) := \pi(\alpha, -1) = (\alpha - 1, -\alpha), (s_2, p_2) := \pi(\alpha, 1) = (\alpha + 1, \alpha)$. Then $C = 1$ and 

$$
\rho(\alpha, \mu_\varepsilon) \leq \frac{1}{1 - \varepsilon}.
$$

Hence $((1 - \varepsilon)/\alpha, (1 - \varepsilon)^2\mu_\varepsilon) \in \mathbb{C}_2$ and so, by Lemma 1.4.3,

$$
|(1 - \varepsilon)\alpha - (1 - \varepsilon)^2\mu_\varepsilon(1 - \varepsilon)|^2 + (1 - \varepsilon)^4|\mu_\varepsilon|^2 \leq 1,
$$

$\alpha \in \partial E$.

It follows that 

$$
(1 - \varepsilon)^3 + (1 - \varepsilon)^4|\mu_\varepsilon|^2 \leq 1
$$

and, finally, $|\mu_\varepsilon| \leq \frac{1 - \varepsilon}{(1 - \varepsilon)^3} \to 0$. \hfill \Box

**Lemma 1.4.11** ([Cos 2003b], [Cos 2004]). Let $\varphi : E \to \mathbb{G}_2$ be a mapping of the form

$$
\varphi = (S, P) = \left( \frac{\tilde{S}}{P_0}, \frac{\tilde{P}}{P_0} \right),
$$

(1.4.6)

where $P_0, \tilde{P}, \tilde{S}$ are polynomials of degree $\leq 2$ with $P_0^{-1}(0) \cap E = \emptyset$. Assume that $\varphi(\partial E) \subset \sigma_2$ and $\varphi(\xi) = (2\eta, \eta^2)$ for some $\xi, \eta \in \partial E$. Then $h := P_0 \circ \varphi \in \text{Aut}(E)$. In
particular, if \( a' := \varphi(\lambda') \), \( a'' := \varphi(\lambda'') \), then
\[
m_E(\lambda', \lambda'') = m_E(h(\lambda'), h(\lambda'')) = m_E(F(\tau(a'), F(\tau(a'')))
\leq \max\{m_E(F(\tau(a'), F(\tau(a''))) : z \in \partial E\}
\leq \max\{m_E(F(\tau(a'), F(\tau(a''))) : z \in \overline{E}\} \leq \kappa^{s_{G_2}}(\lambda', \lambda'') = m_E(\lambda', \lambda'').
\]

Consequently, the formulas from Theorem 1.4.1 hold for all \((s_1, p_1), (s_2, p_2) \in \varphi(E)\), and \( \varphi \) is a complex geodesic.

Proof. Put
\[
h := F_\tau \circ \varphi = \frac{2\pi P - S}{2 - \eta S} = \frac{2\tilde{P} - \tilde{S}}{2\tilde{P}_0 - \tilde{S}_0}.
\]
First observe that \( h(E) \subset E \) and \( h(\partial E) \subset \partial E \). It is clear that \( h \) is a rational function of degree \( \leq 2 \). Notice that \( 2\pi P(\eta) - S(\eta) = 0 = 2\pi P(\eta) - \eta S(\eta) \). Consequently, \( h \) is a rational function of degree \( \leq 1 \) and, therefore, \( h \) must be an automorphism of the unit disc.

Lemma 1.4.12. If \( \varphi \) satisfies the assumptions of Lemma 1.4.11, then for any \( g \in \text{Aut}(E) \), the mapping \( \psi := H_g \circ \varphi \) satisfies the same assumptions.

Proof. The only problem is to check that \( \psi \) has the form (1.4.6). Let \( g = \tau h_a \) for some \( \tau \in \partial E, a \in E \). Fix a \( \lambda \) and let \( \varphi(\lambda) = (S(\lambda), P(\lambda)) = (\pi(z_1, z_2)) \).

\[
\psi(\lambda) = \pi(g(z_1), g(z_2)) = (\tau(h_a(z_1) + h_a(z_2)), \tau^2 h(z_1)h(z_2))
= \left(\tau \frac{(1 + |a|^2)(z_1 + z_2) - 2\pi z_1 z_2 - 2a}{1 - \pi(z_1 + z_2) + \pi^2 z_1 z_2}, \frac{\tau^2 z_1 z_2 - a(z_1 + z_2) + a^2}{1 - \pi(z_1 + z_2) + \pi^2 z_1 z_2}\right).
\]

Consequently,
\[
\psi = \left(\frac{(1 + |a|^2)S - 2\pi P - 2a}{1 - \pi S + \pi^2 P}, \frac{P - aS + a^2}{1 - \pi S + \pi^2 P}\right)
= \left(\frac{\tau(1 + |a|^2)\tilde{S} - 2\pi \tilde{P} - 2ap_0}{p_0 - \pi \tilde{S} + \pi^2 \tilde{P}}, \frac{\tau^2 \tilde{P} - a\tilde{S} + a^2 p_0}{p_0 - \pi \tilde{S} + \pi^2 \tilde{P}}\right). \quad \square
\]

Proof of Theorem 1.4.1. We already know (Lemma 1.4.10) that \( G_2 \) cannot be exhausted by domains biholomorphic to convex domains.

Step 1. First consider the case \( s_1 = 0 \). The case \( s_2 = 0 \) is simple. Consider the embedding \( E \ni \lambda \xrightarrow{\varphi} (0, \lambda) \in G_2 \) and the projection \( G_2 \ni (s, p) \xrightarrow{F} p \in E \). Then
\[
m_E(p_1, p_2) = \max\{m_E(zp_1, zp_2) : z \in \partial E\} = \max\{m_E(zp_1, zp_2) : z \in \overline{E}\}
= m_E(F(0, p_1), F(0, p_2)) \leq \kappa^{s_{G_2}}((0, p_1), (0, p_2))
\leq \tilde{k}^{s_{G_2}}((0, p_1), (0, p_2)) = \tilde{k}^{s_{G_2}}(\varphi(p_1), \varphi(p_2)) \leq m_E(p_1, p_2),
\]
which completes the proof.
Step 2. Assume that \( s_1 = 0, s_2 \neq 0 \). Let \( t_0 \in (0, 1) \) be defined by the formula
\[
t_0 := \max \left\{ m_E \left( p_1, \frac{2p_2 - zs_2}{2 - zs_2} \right) : z \in \partial E \right\} = m_E \left( p_1, \frac{2p_2 - \xi s_2}{2 - \xi s_2} \right),
\]
where \( \xi \in \partial E \).

Our aim is to construct a mapping \( \varphi : E \to \mathbb{G}_2 \) satisfying all the assumptions of Lemma 1.4.11 such that \( \varphi(t_0) = (0, p_2), \varphi(0) = (s_2, p_2) \).

First, we prove that
\[
m_E \left( p_1, \frac{2p_2 - zs_2}{2 - zs_2} \right) < t_0, \quad z \in E,
\]
and so \( \xi \in \partial E \).

Indeed, let \( L : \mathbb{C} \to \mathbb{C}, L(z) := p_1 z - \bar{z} \). Then \( L \) is an \( \mathbb{R} \)-linear isomorphism. In particular, if \( D := L(E) \), then \( \partial D = L(\partial E) \). Observe that
\[
t_0 = \max \{ |\varphi(L(z))| : z \in \partial E \} = \max \{ |\varphi(w)| : w \in \partial D \},
\]
where
\[
\varphi(w) := \frac{2(p_2 - p_1) + s_2 w}{2(1 - p_1 p_2) + s_2 w}.
\]

Note that \( \varphi \neq \text{const} \). Now, the required result follows easily from the maximum principle.

In particular, \( m_E(p_1, p_2) < t_0 \).

Using the automorphism \( R \), we may reduce the problem to the case \( \xi = 1 \).

Step 3. Put
\[
a_0 := F_1(s_2, p_2) = \frac{2p_2 - s_2}{2 - s_2} \in E
\]
and let \( h \in \text{Aut}(E) \) be such that \( h(t_0) = p_1, h(0) = a_0 \). Let \( \tau \in \partial E \) be such that \( h(\tau) = 1 \).

There exists a Blaschke product \( P \) of order 2 such that \( P(t_0) = p_1, P(0) = p_2, \) and \( P(\tau) = 1 \).

Indeed, first observe that it suffices to find a Blaschke product \( Q \) of order 2 with \( Q(t_0) = h_{p_2}(p_1) = p_1', Q(0) = 0, \) and \( Q(\tau) = h_{p_2}(1) =: \tau' \in \partial E \) (having such a \( Q \) we put \( P := h_{-p_2} \circ Q \)).

We have \( Q(\lambda) = \lambda g(\lambda) \), where \( g \in \text{Aut}(E) \) is such that \( g(t_0) = p_1'/t_0 =: a \in E \) (recall that \( t_0 > m_E(p_1, p_2) = |p_1'| \)) and \( g(\tau) = \tau'/\tau =: \tau'' \in \partial E \). Define
\[
g := h_{-a} \circ (\zeta \cdot h_{t_0}),
\]
where \( \zeta := h_a(\tau'') h_{t_0}(\tau) \). Then \( g(t_0) = h_{-a}(0) = a \) and \( g(\tau) = h_{-a}(\zeta \cdot h_{t_0}(\tau)) = \tau'' \).

Step 4. Define
\[
S := \frac{P - h}{1 - h}, \quad \varphi := (S, P).
\]
First observe that $\varphi$ has the form (1.4.6). Indeed, let $P = \tilde{P}/P_0$. The only problem is to show that $S = \tilde{S}/P_0$ with $\tilde{S}$ being a polynomial of degree $\leq 2$. Let $h = \tilde{h}/h_0$. Then

$$S = \frac{\tilde{P}h_0 - \tilde{h}P_0}{P_0(h_0 - h)}.$$ 

Since $h(\tau) = P(\tau) = 1$, the polynomial $\tilde{P}h_0 - \tilde{h}P_0$ is divisible by $\tilde{h} - h_0$.

Observe that:

- $\varphi(t_0) = (S(t_0), P(t_0)) = \left(\frac{2P(t_0) - h(t_0)}{1 - h(t_0)}, p_1\right) = (0, p_1) = (s_1, p_1)$;

- $\varphi(0) = (S(0), P(0)) = \left(\frac{2p_2 - a_0}{1 - a_0}, p_2\right) = \left(\frac{2p_2 - 2p_2 - s_2}{1 - 2p_2 - s_2}, p_2\right) = (s_2, p_2)$;

- $F_1 \circ \varphi = \frac{2P - S}{2 - S} = \frac{2P - 2P/h}{2 - 2h/h} = h$;

- on $\partial E$ we get

  $$\Im P = \frac{2P - \tilde{h}}{1 - h} P = \frac{2(1 - P/h)}{1 - h} = S.$$ 

We have

$$h = \frac{2P - S}{2 - S} = \frac{2\tilde{P} - \tilde{S}}{2P_0 - \tilde{S}}.$$ 

Note that $\deg(2\tilde{P} - \tilde{S}) = 2$ or $\deg(2P_0 - \tilde{S}) = 2$. Thus the polynomials $2\tilde{P} - \tilde{S}$, $2P_0 - \tilde{S}$ must have a common zero, say $z_0$. We have $2\tilde{P}(z_0) = \tilde{S}(z_0) = 2P_0(z_0)$. Thus $P(z_0) = 1$, which implies that $z_0 \in \partial E$ and $S(z_0) = 2$.

Put $C := \max\{\Im(\lambda) : \lambda \in \partial E\}$ (we already know that $C \geq 2$). Define $\psi := (2S/C, P)$. Then $\psi$ satisfies all the assumptions of Lemma 1.4.11 and, consequently, Theorem 1.4.1 holds for points from $\psi(E)$. In particular, there exists an $\eta \in \overline{E}$ such that

$$t_0 = m_E(p_1, \frac{2\eta p_2 - 2s_2/C}{2 - \eta 2s_2/C}) = m_E(p_1, \frac{2p_2 - \eta 2s_2/C}{2 - \eta 2s_2/C}).$$

Hence, $C \leq 2$ and finally $C = 2$. Consequently, $\varphi = \psi$, which completes the proof of the theorem in the case $s_1 = 0$.

Step 5. Now let $(s_1, p_1), (s_2, p_2) \in G_2$ be arbitrary. Suppose that $s_1 = \lambda_1^0 + \lambda_2^0$ with $\lambda_1^0, \lambda_2^0 \in E$. One can easily prove that there exists an automorphism $g \in \text{Aut}(E)$ such that $g(\lambda_1^0) + g(\lambda_2^0) = 0$. Then $H_2(s_1, p_1) = (0, p_1')$. Put $(s_2', p_2') := H_2(s_2, p_2)$. We have the following two cases:

- $s_2' = 0$: We already know that the mapping $\varphi = (0, h)$ with suitable $h \in \text{Aut}(E)$ ($h(t_0) = p_1', t_0 := m_E(p_1', p_2'), h(0) = p_2'$) is a complex geodesic for $(0, p_1')$ and $(0, p_2')$. 

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Corollary 1.4.13.

By an argument like in the proof of Lemma 1.4.12, we easily conclude that if $g^{-1} = \tau h_a$, then
\[
\psi := H^{-1} \circ \varphi = \left( \tau - \frac{2\pi h - 2a}{1 + \overline{a} h}, \tau^2 \frac{h + a^2}{1 + \overline{a} h} \right) = (\overline{\beta} q + \beta, q),
\]
where
\[
\beta := -\tau \frac{2a}{1 + |a|^2} \in E, \quad q := \tau^2 h - a^2 \circ h \in \text{Aut}(E).
\]

For any $\alpha \in \partial E$ we have:
\[
F_\alpha \circ \psi = \frac{2q - \overline{\pi}(\overline{\beta} q + \beta)}{2 - \alpha(\overline{\beta} q + \beta)} = \frac{2 - \pi \overline{\beta}}{2 + \alpha \overline{\beta}} \cdot \frac{q - \frac{\pi \beta}{2 - \pi \overline{\beta}}}{1 - \frac{\alpha^2}{\overline{\alpha}}} =: q_\alpha \in \text{Aut}(E).
\]
Hence
\[
t_0 = m_E(q_\alpha(t_0), q_\alpha(0)) = m_E(F_\alpha(\psi(t_0)), F_\alpha(\psi(0)))
\leq \max \left\{ m_E\left( \frac{2p_1 - \overline{s_1}}{2 - \overline{s_1}}, \frac{2p_2 - \overline{s_2}}{2 - \overline{s_2}} \right) : z \in \partial E \right\}
= \max \{ m_E(F_2(s_1, p_1), F_2(s_2, p_2)) : z \in \partial E \}
\leq \max \{ m_E(F_2(s_1, p_1), F_2(s_2, p_2)) : z \in \overline{E} \}
\leq c_{G_2}^*(\langle s_1, p_1 \rangle, \langle s_2, p_2 \rangle) \leq \tilde{k}_{\mathbb{G}_2}^*(\langle s_1, p_1 \rangle, (s_2, p_2)) = \tilde{k}_{\mathbb{G}_2}^*(\psi(t_0), \psi(0)) \leq t_0.
\]

\* $s'_2 \neq 0$: We know that there exists a mapping $\varphi : E \rightarrow \mathbb{G}_2$ as in Lemma 1.4.11 such that $H_2(s_j, p_j) \in \varphi(E)$, $j = 1, 2$. It remains to observe that, by Lemma 1.4.12, the mapping $H_2^{-1} \circ \varphi$ also satisfies all the assumptions of Lemma 1.4.11. \qed

Corollary 1.4.13.

\[
c_{G_2}^*\left( \langle s_1, p_1 \rangle, \langle s_2, p_2 \rangle \right) = \tilde{k}_{\mathbb{G}_2}^*\left( \langle s_1, p_1 \rangle, \langle s_2, p_2 \rangle \right)
\max \left\{ \left| (s_1 p_2 - p_1 s_2) z^2 + 2(p_1 - p_2) z + s_2 - s_1 \right| : z \in \partial E \right\}, \quad (s_1, p_1), (s_2, p_2) \in \mathbb{G}_2.
\]

In particular,
\[
c_{G_2}^*\left( \langle 0, 0 \rangle, (s, p) \right) = \tilde{k}_{\mathbb{G}_2}^*\left( \langle 0, 0 \rangle, (s, p) \right) = \max\{|F_2(s, p)| : z \in \partial E\}
\leq \frac{2|s - \overline{p}| + |s^2 - 4p|}{4 - |s|^2}, \quad (s, p) \in \mathbb{G}_2.
\]

Theorem 1.4.14 ([Jar-Pfl 2003b], [Cos 2004]).

\[
\text{Aut}(\mathbb{G}_2) = \{ H_k : h \in \text{Aut}(E) \}. \quad (19)
\]

A characterization of $\text{Aut}(\mathbb{G}_2)$ is also announced in [Agl-You 2003] for a future paper.

\( (19) \) See Remark 1.4.17 for a more general result.
Proof. Step 1. First observe that $\text{Aut}(G_2)$ does not act transitively on $\mathbb{C}^2$.

Otherwise, by the Cartan classification theorem (cf. [Ahk 1990], [Fuk 1965]), $G_2$ would be biholomorphic to $\mathbb{B}_2$ or $E^2$, which is, by Theorem 1.4.1, impossible (20).

Step 2. Next observe that $F(\Sigma_2) = \Sigma_2$ for every $F \in \text{Aut}(G_2)$.

Indeed, let $V := \{ F(0,0) : F \in \text{Aut}(G_2) \}$. By W. Kaups’ theorem, $V$ is a connected complex submanifold of $G_2$ (cf. [Kau 1970]). We already know that $\Sigma_2 \subset V$ (Remark 1.4.8). Since $\text{Aut}(G_2)$ does not act transitively, we have $V \not\subset G_2$. Thus $V = \Sigma_2$.

Take a point $(s_0, p_0) = H_h(0,0) \in \Sigma_2$ with $h \in \text{Aut}(G)$ (Remark 1.4.8). Then for every $F \in \text{Aut}(G_2)$, we get $F(s_0, p_0) = (F \circ H_h)(0,0) \in V = \Sigma_2$.

Step 3. By Remark 1.4.8, we only need to show that every automorphism $F \in \text{Aut}(G_2)$ with $F(0,0) = (0,0)$ is equal to a “rotation” $R_\tau$. Fix such an $F = (S, P)$.

First observe that $F|_{\Sigma_2} \in \text{Aut}(\Sigma_2)$. Hence the mapping

$$E \ni \lambda \longrightarrow (2\lambda, \lambda^2) \longrightarrow F(2\lambda, \lambda^2) \longrightarrow \frac{1}{2} \text{pr}_R(F(2\lambda, \lambda^2)) \in E$$

must be a rotation, i.e. $F(2\lambda, \lambda^2) = (2\alpha \lambda, \alpha^2 \lambda^2)$ for some $\alpha \in \partial E$. Taking $R_{1/\alpha} \circ F$ instead of $F$, we may assume that $\alpha = 1$. In particular, $F'(0,0) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and, therefore,

$F'(0,0) = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$.

For $\tau \in \partial E$ put $G_\tau := F^{-1} \circ R_{1/\tau} \circ F \circ R_\tau \in \text{Aut}(G_2)$. Obviously, $G_\tau(0,0) = (0,0)$.

Moreover, $G_\tau'(0,0) = \begin{bmatrix} 1 & b(\tau - 1) \\ 0 & 1 \end{bmatrix}$. Let $G_\tau^n : G_2 \longrightarrow G_2$ be the $n$–th iterate of $G_\tau$. We have $(G_\tau^n)'(0,0) = \begin{bmatrix} 1 & nb(\tau - 1) \\ 0 & 1 \end{bmatrix}$. Using the Cauchy inequalities, we get

$$|nb(\tau - 1)| \leq \text{const}, \quad n \in \mathbb{N}, \quad \tau \in \partial E,$$

which implies that $b = 0$, i.e. $F'(0,0)$ is diagonal.

Step 4. We have $G_\tau'(0,0) = E_2$. Hence, by the Cartan theorem (cf. [Nar 1971], p. 66), $G_\tau = \text{id}$. Consequently, $R_\tau \circ F = F \circ R_\tau$, i.e.

$$(\tau S(s, p), \tau^2 P(s, p)) = (S(\tau s, \tau^2 p), P(\tau s, \tau^2 p)), \quad (s, p) \in G_2, \quad \tau \in \partial E.$$ 

Hence $F(s, p) = (s, p + Cs^2)$. Since $F(2\lambda, \lambda^2) = (2\lambda, \lambda^2)$, we have $(2\lambda, \lambda^2 + 4C\lambda^2) = (2\lambda, \lambda^2)$, which immediately implies that $C = 0$, i.e. $F = \text{id}$. \hfill \Box

By virtue of Theorem 1.4.1 we know that for any two points $(s_1, p_1), (s_2, p_2) \in G_2$ there exists a complex geodesic $\varphi$ with $(s_1, p_1), (s_2, p_2) \in \varphi(E)$ (cf. §1.2.5). Moreover, there exists an $\alpha \in \partial E$ such that $F_\alpha \circ \varphi \in \text{Aut}(E)$. We have the following characterization of complex geodesics in $G_2$, cf. [Pfl-Zwo 2003b].

\(^{(20)}\) Instead of Theorem 1.4.1, one can also argue as follows: In the case $G_2 \simeq B_2$ we use the Remmert–Stein theorem (cf. [Nar 1971], p. 71) saying that there is no proper holomorphic mapping $E^2 \longrightarrow B_2$. In the case $G_2 \simeq E^2$ we use the characterization of proper holomorphic mappings $F : E^2 \longrightarrow E^2$ (cf. [Nar 1971], p. 76), saying that any such a mapping has the form $F(z_1, z_2) = (F_1(z_1), F_2(z_2))$ up to a permutation of the variables.
Theorem 1.4.15. Let \( \varphi = (S, P) : E \rightarrow \mathbb{G}_2 \). Then:

(a) If \#(\varphi(E) \cap \Sigma_2) \geq 2, then \( \varphi \) is a complex geodesic iff \( \varphi(\lambda) = (-2\lambda, \lambda^2) \) \( (\lambda \in E) \) mod \( \text{Aut}(E) \). In particular, if \( \varphi \) is a complex geodesic, then \( \varphi(E) = \Sigma_2 \).

(b) If \#(\varphi(E) \cap \Sigma_2) = 1, then \( \varphi \) is a complex geodesic iff \( \varphi(\lambda) = \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda})) \) \( (\lambda \in E) \) mod \( \text{Aut}(E) \), where \( B \) is a Blaschke product of order \( \leq 2 \) with \( B(0) = 0 \).

(c) If \( \varphi(E) \cap \Sigma_2 = \emptyset \), then \( \varphi \) is a complex geodesic iff \( \varphi = \pi(h_1, h_2) \), where \( h_1, h_2 \in \text{Aut}(E) \) are such that \( h_1 \cdot h_2 \) has no zero in \( E \).

In particular, any complex geodesics \( \varphi : E \rightarrow \mathbb{G}_2 \) extends holomorphically to \( \overline{E} \) and \( \varphi(\partial E) \subseteq \sigma_2 \).

A concrete description of all complex geodesics in \( \mathbb{G}_2 \) is also announced in [Agl-You 2003] for a future paper.

Proof. (a) Let \( \varphi(\lambda) := (-2\lambda, \lambda^2), \lambda \in E \). Then \( \varphi(E) = \Sigma_2 \). Consequently, by Lemma 1.4.11, \( \varphi \) is a complex geodesic.

Now, let \( \varphi : E \rightarrow \mathbb{G}_2 \) be a complex geodesic with \( \varphi(\xi) = (2\mu, \mu^2) \in \Sigma_2 \) \( (\xi, \mu \in E) \). Taking \( \varphi \circ h_{-\xi} \) instead of \( \varphi \), we may assume that \( \xi = 0 \). Taking \( H_{h_{\xi}} \circ \varphi \) instead of \( \varphi \), we may assume that \( \mu = 0 \), i.e. \( \varphi(0, 0) = (0, 0) \). By Theorem 1.4.1, there exists an \( \alpha \in \partial D \) such that \( F_\alpha \circ \varphi = \frac{2P-P(\alpha)S}{2-\alpha S} \circ h \in \text{Aut}(E) \). Taking \( R_\alpha \circ \varphi \) instead of \( \varphi \) we may assume that \( \alpha = 1 \). Observe that \( h \) must be a rotation and, therefore, we may also assume that \( h = \text{id} \), i.e. \( \frac{2P-P(\alpha)S}{2-\alpha S} = \text{id} \). Thus \( S(\lambda) = 2P(\lambda) = \lambda, \lambda \in E \).

Let \( \eta \in E_* \) be such that \( \varphi(\eta) \in \Sigma_2 \). Then \( S^2(\eta) = 4P(\eta) \), i.e. \( \left( \frac{P(\eta)-\eta}{1-\eta} \right)^2 = P(\eta) \).

Hence \( P(\eta) = \eta^2 \) and so \( P(\eta) = -2\eta \). The Schwarz lemma implies that \( S(\lambda) = -2\lambda, \lambda \in E \), and, finally, \( P(\lambda) = \lambda^2, \lambda \in E \).

(b) Let \( \varphi(\lambda) := \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda})), \lambda \in E \), where \( B \) is a Blaschke product of order \( \leq 2 \) with \( B(0) = 0 \).

In the case \( B(\lambda) = \tau \lambda, \lambda \in E \) \( (\tau \in \partial E) \), we get \( \varphi(\lambda) = (0, -\tau^2 \lambda), \lambda \in E \). Consequently, \( F_\alpha \circ \varphi \in \text{Aut}(E) \) for any \( \alpha \in \partial E \) (cf. Step 5 of the proof of Theorem 1.4.1).

In the case \( B(\lambda) = \tau \lambda h_\xi(\lambda), \lambda \in E \) \( (\tau \in \partial E, b \in E) \), we get \( \varphi(\partial E) \subseteq \sigma_2 \) and

\[
\varphi(\lambda) = \left( 2\tau \lambda \frac{1-|b|^2}{1-\bar{b}\lambda}, \tau^2 \lambda \frac{\lambda-b^2}{1-b^2 \lambda} \right), \quad \lambda \in E.
\]

To apply Lemma 1.4.11, we only need to observe that \( \varphi(\xi) = (2\mu, \mu^2) \) for some \( \xi, \mu \in \partial E \). The case \( b = 0 \) is obvious. In the case \( b \neq 0 \) take \( \xi := \frac{b}{\bar{b}} \). Then \( \varphi(\xi) = (2\tau \xi, \tau^2 \xi^2) \).

Now, let \( \varphi : E \rightarrow \mathbb{G}_2 \) be a complex geodesic with \( \#(\varphi(E) \cap \Sigma_2) = 1 \). Then, as in (a), we may assume that \( \varphi(0, 0) = (0, 0), \frac{2P-P(\alpha)S}{2-\alpha S} = \text{id} \). Observe that \( \Delta(\lambda) := S^2(\lambda) - 4P(\lambda) \neq 0 \) for \( \lambda \in E_* \). Write \( \Delta(\lambda) = \lambda^k \tilde{\Delta}(\lambda) \), where \( \tilde{\Delta}(\lambda) \neq 0, \lambda \in E \). Define

\[
B(\lambda) := \frac{1}{2} \left( S(\lambda^2) + \lambda^k \sqrt{\tilde{\Delta}(\lambda^2)} \right), \quad \lambda \in E.
\]

Then

\[
S^2(\lambda^2) - 4P(\lambda^2) = \Delta(\lambda^2) = \lambda^{2k} \tilde{\Delta}(\lambda^2) = 4B^2(\lambda) - 4B(\lambda)S(\lambda^2) + S^2(\lambda^2), \quad \lambda \in E,
\]
which implies that 

\[ B(\lambda)S(\lambda^2) - B^2(\lambda) = P(\lambda^2) = B(-\lambda)S(\lambda^2) - B^2(-\lambda) \]

and, consequently,

\[ (B(\lambda) - B(-\lambda))(S(\lambda^2) - (B(\lambda) + B(-\lambda))) = 0, \quad \lambda \in E. \]

We have the following two cases:

(i) \( S(\lambda^2) = B(\lambda) + B(-\lambda), \lambda \in E \):

Then

\[ P(\lambda^2) = B(\lambda)S(\lambda^2) - B^2(\lambda) = B(\lambda)(B(\lambda) + B(-\lambda)) - B^2(\lambda) = B(\lambda)B(-\lambda), \quad \lambda \in E. \]

Hence \( \varphi(\lambda) = \pi(B(\lambda)), B(-\lambda), \lambda \in E. \)

Fix \( t_0 \in (0, 1) \). Let \( (s_0, p_0) := \varphi(t_0) = \pi(B(t_0), B(-t_0)) = \pi(0^0, 0^0). \)

Suppose that there exists a function \( f \in \mathcal{O}(E, E) \) such that \( f(0) = 0, f(t_0) = \lambda_0^0, f(-t_0) = \lambda_0^0, \) and \( f(E) \in E \). Put \( \psi := \pi(f, f) : E \rightarrow \mathbb{G}_2 \) and observe that \( \psi(0) = (0, 0) \) and \( \psi(t_0^0) = (s_0, p_0) \). Hence \( \psi \) would be a complex geodesic with \( \psi(E) \in \mathbb{G}_2 \) — contradiction.

Thus, the function \( B \) solves an extremal problem of 2–type in the sense of [Edi 1995]. Consequently, \( B \) must be a Blaschke product of order \( \leq 2 \).

(ii) \( B(\lambda) = B(-\lambda), \lambda \in E \): Then there exists a function \( B_1 \in \mathcal{O}(E, E) \) such that

\[ B(\lambda) = B_1(\lambda^2) = \frac{1}{2} \left( S(\lambda^2) + \lambda^k \sqrt{\Delta(\lambda^2)} \right), \quad \lambda \in E. \]

Using the same argument, we reduce the proof to the case, where there exists a \( B_2 \in \mathcal{O}(E, E) \) such that

\[ B_2(\lambda^2) = \frac{1}{2} \left( S(\lambda^2) - \lambda^k \sqrt{\Delta(\lambda^2)} \right), \quad \lambda \in E. \]

Hence \( \varphi = \pi(B_1, B_2) \). Since \( \frac{2P - S}{2 - S} = \text{id} \), we get

\[ 2B_1(\lambda)B_2(\lambda) - (B_1(\lambda) + B_2(\lambda)) = \lambda(2 - (B_1(\lambda) + B_2(\lambda))), \quad \lambda \in E, \]

which gives \((B_1'(0) + B_2'(0)) = 2 \). Consequently, by the Schwarz lemma, \( B_1(\lambda) = B_2(\lambda) = -\lambda, \) and finally \( \varphi(\lambda) = (-2\lambda, \lambda^2), \lambda \in E. \) Thus, \( \Delta \equiv 0 \); contradiction.

(c) Let \( \varphi := \pi(h_1, h_2) \), where \( h_1, h_2 \in \text{Aut}(E) \) are such that \( h_1 - h_2 \) has no zero in \( E \). Observe that \( \varphi \) satisfies (1.4.6) and \( \varphi(\partial E) \subset \sigma_2 \). To use Lemma 1.4.11 we only need to check that \( \varphi(\xi) = (2\eta, \eta^2) \) for some \( \xi, \eta \in \partial E \), i.e. \( h_1(\xi) = h_2(\xi) \) for some \( \xi \in \partial E \). Let \( h_j = \tau_j h_{a_j} \ (\tau_j \in \partial E, a_j \in E), j = 1, 2 \). Then we have to find a root \( z = \xi \) of the equation

\[ \tau_1(z - a_1)(1 - \overline{\tau_2}z) - \tau_2(z - a_2)(1 - \overline{\tau_1}z) = A_2 z^2 + A_1 z + A_0 = 0 \]

with \( |\xi| = 1 \). We have \( A_2 = -\tau_1 \overline{\tau_2} + \tau_2 \overline{\tau_1}, A_0 = -\tau_1 a_1 + \tau_2 a_2 \). Observe that \( |A_2| = |A_0| \).

Since the equation \( h_1 - h_2 = 0 \) has no roots in \( E \), we get \( A_2 \neq 0 \). Let \( z_1, z_2 \) be the roots of the above equation. We have \( |z_1|, |z_2| \geq 1 \) and \( |z_1 z_2| = |A_0/A_2| = 1 \). Thus \( |z_1| = |z_2| = 1 \).

Now, let \( \varphi : E \rightarrow \mathbb{G}_2 \) be a complex geodesic with \( \varphi(E) \cap \Sigma_2 = \emptyset \). Then there exists a holomorphic mapping \( \psi : E \rightarrow E^2 \) with \( \pi \circ \psi = \varphi \). Consequently, \( \psi \) must be a complex
geodesic \( m_E(\lambda', \lambda'') = c_{E_2}^g(\varphi(\lambda'), \varphi(\lambda'')) \leq c_{E_2}^g(\psi(\lambda'), \psi(\lambda'')) \leq m_E(\lambda', \lambda'') \). Hence, 
\( \psi = (h_1, h_2) \), where \( h_1, h_2 \in \mathcal{O}(E, E) \) and at least one of \( h_1 \) and \( h_2 \) is an automorphism. Assume that \( h_1 \in \text{Aut}(E) \).

Fix a \( t_0 \in (0, 1) \) and suppose that \( m_E(h_2(0), h_2(t_0)) < t_0 \). Let

\[
\rho := m_E(h_2(0), h_2(t_0))/t_0 \in (0, 1).
\]

There exists a \( g \in \text{Aut}(E) \) such that \( g(0) = h_2(0) \), \( g(\rho t_0) = h_2(t_0) \). Put \( f(\lambda) := g(\rho \lambda) \), \( \lambda \in \mathcal{E} \). Then \( f(0) = h_2(0) \), \( f(t_0) = h_2(t_0) \), and \( f(E) \subseteq E \). Put \( \chi = (\chi_1, \chi_2) := \pi(h_1, f) \). Then \( \chi(0) = \varphi(0) \) and \( \chi(t_0) = \varphi(t_0) \). Thus \( \chi \) is also a complex geodesic. Notice that by the Rouché theorem the function \( h_1 - f \) has a zero in \( E \). Hence \( \chi(E) \cap \Sigma_2 \neq \emptyset \). In particular, in view of (a) and (b), \( \chi(\partial E) \subseteq \sigma_2 \). On the other hand \( \chi_2 = h_1 f \); contradiction.

Consequently, \( m_E(h_2(0), h_2(t_0)) = t_0 \), and, therefore, \( h_2 \in \text{Aut}(E) \).

\[ \square \]

**Remark 1.4.16.** With the help of Theorem 1.4.1, the Carathéodory and Kobayashi pseudodistances and the Lempert function were calculated for the following unbounded balanced domain

\[
\Omega_2 := \{ A \in \mathbb{C}(2 \times 2) : r(A) < 1 \},
\]

where \( r(a) \) denotes the spectral radius of \( A \); cf. [Cos 2004].

**Remark 1.4.17.** Let \( n \geq 3 \) and let \( \pi_n : \mathbb{C}^n \longrightarrow \mathbb{C}^n \),

\[
\pi_n(\lambda_1, \ldots, \lambda_n) := \left( \sum_{1 \leq j_1 < \cdots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k} \right)_{k=1, \ldots, n}
\]

Observe that \( \pi_n, \pi_n|_{E^n} \) are proper. Put \( \mathcal{G}_n := \pi_n(E^n) \). The domain \( \mathcal{G}_n \) is called the symmetrized \( n \)-disc.

Recently, A. Edigarian and W. Zwonek proved in [Edi-Zwo 2004] the following result.

**Theorem.** Any proper holomorphic mapping \( F : \mathcal{G}_n \longrightarrow \mathcal{G}_n \) is of the form

\[
F((\pi_n(\lambda_1, \ldots, \lambda_n)) = \pi_n(B(\lambda_1), \ldots, B(\lambda_n)),
\]

where \( B \) is a finite Blaschke product. In particular,

\[
\text{Aut}(\mathcal{G}_n) = \{ H_h : h \in \text{Aut}(E) \},
\]

where \( H_h(\pi_n(\lambda_1, \ldots, \lambda_n)) = \pi_n(h(\lambda_1), \ldots, h(\lambda_n)) \).

It is an open question whether:

\[ \Box \] \( c_{n}^\mathcal{G}_n \equiv k_{n}^\mathcal{G}_n \), i.e. \( \mathcal{G}_n \in \mathcal{L}_n \) ?

\[ \Box \] \( \mathcal{G}_n \) cannot be exhausted by domains biholomorphic to convex domains ?

Moreover, one can conjecture that for any proper mapping \( F : \mathbb{C}^n \longrightarrow \mathbb{C}^n \) the domain \( \mathcal{G} := F(E^n) \) belongs to \( \mathcal{L}_n \).
1.5. Generalized holomorphically contractible families

Observe that the Möbius and Lempert functions are obviously symmetric ($c_G^c$ is even a pseudodistance). The higher Möbius functions and the Green function are in general not symmetric (cf. [J-P 1993], § 4.2). Their definitions distinguish one point (pole) at which we impose growth conditions. From that point of view it is natural to investigate objects with more general growth conditions. For instance, the Green function $g_G$ may be generalized as follows.

**Definition 1.5.1.** Let $G \subset \mathbb{C}^n$ be a domain and let $p : G \rightarrow \mathbb{R}_+$ be a function. Define

$$g_G(p, z) := \sup \{ u(z) : u : G \rightarrow [0, 1], \log u \in \mathcal{P}\mathcal{S}\mathcal{H}(G),$$

$$\forall a \in G, \exists C = \mathcal{C}(u, a) > 0 \forall w \in G : u(w) \leq C \| w - a \|^p \langle a \rangle, \quad z \in G.$$ \hspace{1cm} (21)

The function $g_G(p, \cdot)$ is called the *generalized pluricomplex Green function with poles* (weights, pole function) $p$.

We have $g_G(0, \cdot) \equiv 1$. Observe that if the set $|p| := \{ z \in G : p(z) > 0 \}$ is not pluripolar, then $g_G(p, \cdot) \equiv 0$. Obviously, $g_G(p, z) = 0$ for every $z \in |p|$.

In the case where $p = \chi_A = \text{the characteristic function of a set } A \subset G$, we put $g_G(A, \cdot) := g_G(\chi_A, \cdot)$. Obviously, $g_G(\{a\}, \cdot) = g_G(a, \cdot), a \in G$. In the case where the set $|p|$ is finite, the function $g_G(p, \cdot)$ was introduced by P. Lelong in [Le 1989].

The definition of the generalized Green function may be formally extended to the case where $p : G \rightarrow [0, +\infty)$. We put $g_G(p, \cdot) \equiv 0$ if there exists a $z_0 \in G$ with $p(z_0) = +\infty$.

The generalized pluricomplex Green function was recently studied by many authors, e.g. [Car-Wie 2003], [Com 2000], [Edi 2002], [Edi-Zwo 1998b], [Jar-Jar-Pfl 2003], [Lár-Sig 1998b].

Using similar ideas, one can generalize the Möbius function.

**Definition 1.5.2.** Let $G \subset \mathbb{C}^n$ be a domain and let $p : G \rightarrow \mathbb{Z}_+$ be a function. Define

$$m_G(p, z) := \sup \{ f(z) : f \in \mathcal{O}(G, E), \text{ord}_a f \geq p(a), a \in G \}, \quad z \in G.$$ The function $m_G(p, \cdot)$ is called the *generalized Möbius function with weights* $p$.

We have $m_G(0, \cdot) \equiv 1$. Notice that if the set $|p|$ is not thin, then $m_G(p, \cdot) \equiv 0$. As before the definition may be formally extended to the case where $p : G \rightarrow \mathbb{Z}_+ \cup \{+\infty\}$; $m_G(p, \cdot) \equiv 0$ if there exists a $z_0 \in G$ with $p(z_0) = +\infty$. Similarly as in the case of the generalized Green function, we put $m_G(A, \cdot) := m_G(\chi_A, \cdot) (A \subset G)$, $m_G(a, \cdot) := m_G(\{a\}, \cdot) (a \in G)$ \hspace{1cm} (22).

Obviously, $m_G(a, \cdot) = c_G^c(a, \cdot), a \in G$. More generally, $m_G(k\chi(a), \cdot) = [m_G^k(a, \cdot)]^k$. It is clear that $m_G(p, \cdot) \leq g_G(p, \cdot)$ (for any function $p : G \rightarrow \mathbb{Z}_+ \cup \{+\infty\}$). Properties of $g_G(p, \cdot)$ and $m_G(p, \cdot)$ will be presented in § 1.6.

---

(21) Here and in the sequel $0^0 := 1$. Observe that the condition is trivially satisfied at all points $a \in G$ with $p(a) = 0$. The growth condition may be equivalently formulated as follows:

$$\forall a \in G, p(a) > 0, \exists C > 0 : u(w) \leq C \| w - a \|^p \langle a \rangle, w \in B(a, r) \subset G.$$\hspace{1cm} (22) Note that in the case of the unit disc this definition of the function $m_G$ coincides with the previous one from Definition 1.1.1.
The above generalizations lead us to the following definition.

**Definition 1.5.3.** A family \((d_G)_G\) of functions \(d_G : \mathbb{R}_+^G \times G \to \mathbb{R}_+\), where \(\mathbb{R}_+^G\) denotes the family of all functions \(p : G \to \mathbb{R}_+\), is said to be a **generalized holomorphically contractible family** (g.h.c.f.) if the following three conditions are satisfied:

\[(E) \text{ } \prod_{a \in E} \{m_E(a, z)\}^{p(a)} \leq d_E(p, z) \leq \inf_{a \in E} \{m_E(a, z)\}^{p(a)}, \quad (p, z) \in \mathbb{R}_+^E \times E \quad (23),\]

\[(H) \text{ } \text{for any } F \in \mathcal{O}(G, D) \text{ and } q : D \to \mathbb{R}_+, \text{ we have}\]

\[d_D(q, F(z)) \leq d_G(q \circ F, z), \quad z \in G,\]

\[(M) \text{ } \text{for any } p, q : G \to \mathbb{R}_+, \text{ if } p \leq q, \text{ then } d_G(q, \cdot) \leq d_G(p, \cdot).\]

If in the above definition one considers only integer-valued weights (like in the case of the generalized Möbius function), then we get the definition of a **generalized holomorphically contractible family with integer-valued weights**.

Put \(d_G(A, \cdot) := d_G(\chi_A, \cdot) (A \subset G)\), \(d_G(\cdot, z) := d_G(\cdot, \{z\}) (a \in G)\).

One can prove that the generalized Green and Möbius functions are g.h.c.f. in the sense of the above definition; cf. § 1.6. In the context of the inequalities (1.1.2), it is natural to ask whether there exist minimal and maximal g.h.c.f. Put

\[d_G^{\text{min}}(p, z) := \sup \left\{ \prod_{\mu \in f(A)} \{m_E(\mu, f(z))\}^{\sup p(f^{-1}(\mu))} : f \in \mathcal{O}(G, E) \right\} \]

\[= \sup \left\{ \prod_{\mu \in f(G)} |\mu|^{\sup p(f^{-1}(\mu))} : f \in \mathcal{O}(G, E), \ f(z) = 0 \right\}, \quad (24)\]

\[d_G^{\text{max}}(p, z) = \mathbb{K}^*_G(p, z) := \inf \{\mathbb{K}^*_G(a, z) : a \in G\} \]

\[= \inf \{\mathbb{K}^*_G(\varphi, \mu) : \varphi \in \mathcal{O}(E, G), \ \varphi(0) = z, \ \mu \in E\}, \quad z \in G.\]

We have \(d_G^{\text{min}}(0, \cdot) = d_G^{\text{max}}(0, \cdot) \equiv 1\). Observe that \(d_G^{\text{min}}(k\chi_{\{z\}}, \cdot) = [\mathbb{K}^*_G(a, \cdot)]^k\) and \(d_G^{\text{max}}(k\chi_{\{z\}}, \cdot) = [\mathbb{K}^*_G(a, \cdot)]^k\). Moreover, for \(\varnothing \neq A \subset G\) we get

\[d_G^{\text{min}}(A, z) = \sup \left\{ \prod_{\mu \in f(A)} m_E(\mu, f(z)) : f \in \mathcal{O}(G, E) \right\} \]

\[\geq \sup \{\{f(z) : f \in \mathcal{O}(G, E), \ f|_A = 0\} = m_G(A, z), \quad z \in G. \quad (25)\]

We extend formally the definitions of \(d_G^{\text{min}}(p, \cdot)\) and \(d_G^{\text{max}}(p, \cdot)\) to the case where \(p : G \to [0, +\infty]\); \(d_G^{\text{min}}(p, \cdot) = d_G^{\text{max}}(p, \cdot) \equiv 0\) if there exists a \(z_0 \in G\) with \(p(z_0) = +\infty\).

Directly from the definitions it follows that the systems \((d_G^{\text{min}})_G, (d_G^{\text{max}})_G\) satisfy (E) and (M) of Definition 1.5.3.

**Proposition 1.5.4** ([Jar-Jar-Pfl 2003]). The systems \((d_G^{\text{min}})_G\) and \((d_G^{\text{max}})_G\) are g.h.c.f. Moreover, for any g.h.c.f. \((d_G)_G\) (with integer-valued weights) we have

\[d_G^{\text{min}}(p, \cdot) \leq d_G(p, \cdot) \leq d_G^{\text{max}}(p, \cdot)\]

(23) For \(h : A \to [0, 1]\), we put \(\prod_{a \in A} h(a) := \inf_{B \subseteq A} \prod_{a \in B} h(a)\).

(24) Note that if \(\sup p(f^{-1}(\mu_0)) = +\infty\) for a \(\mu_0 \in f(G)\), then \(\prod_{\mu \in f(G)} [m_E(\mu, f(z))]^{\sup p(f^{-1}(\mu))} = 0\).

(25) We will see (cf. Proposition 1.5.4) that in fact \(d_G^{\text{min}}(A, \cdot) = m_G(A, \cdot)\).
for any function \( p : G \to \mathbb{R}_+ \) (\( p : G \to \mathbb{Z}_+ \)). In particular,
\[
d^\min_G(p, \cdot) \leq g_G(p, \cdot) \leq d^\max_G(p, \cdot)
\]
for any function \( p : G \to \mathbb{R}_+ \) and
\[
d^\min_G(p, \cdot) \leq m_G(p, \cdot) \leq g_G(p, \cdot) \leq d^\max_G(p, \cdot)
\]
for any function \( p : G \to \mathbb{Z}_+ \).

Consequently, \( d^\min_G(A, \cdot) = m_G(A, \cdot), A \subset G \).

The function \( d^\min_G \) (resp. \( d^\max_G \)) may be considered as a generalization of the Möbius function \( c^*_G \) (resp. Lempert function \( \tilde{k}^*_G \)). Properties of \( d^\min_G \) and \( d^\max_G \) will be presented in § 1.8.

Proof. Step 1. If \( (d_G)_G \) satisfies (H) and
\[
(E^+) \quad d^E_G(p, \lambda) \leq d^\max_G(p, \lambda) = \inf \{ [m_E(\mu, \lambda)]^{p(\mu)} : \mu \in E \}, (p, \lambda) \in \mathbb{R}_+^E \times E,
\]
then \( d_G \leq d^\max_G \) for any \( G \). The result remains true in the category of g.h.c.f. with integer-valued weights.

Indeed,
\[
d_G(p, z) \overset{(H)}\leq \inf \{ d_E(p \circ \varphi, 0) : \varphi \in O(E, G), \varphi(0) = z \}
\]
\[
\overset{(E^+)}\leq \inf \{ |\mu|^{p(\varphi(\mu))} : \varphi \in O(E, G), \varphi(0) = z, \mu \in E \}
\]
\[
= d^\max_G(p, z), \quad (p, z) \in \mathbb{R}_+^G \times G.
\]

Step 2. The system \( (d^\max_G)_G \) is a g.h.c.f. Indeed, to prove (H) let \( F : G \to D \) be holomorphic and let \( q : D \to \mathbb{R}_+ \). Then
\[
d^\max_D(q, F(z)) = \inf \{ (\tilde{k}^*_D(b, F(z))]^{q(b)} : b \in D \}
\]
\[
\leq \inf \{ (\tilde{k}^*_D(F(a), F(z))]^{q(F(\alpha))} : a \in G \}
\]
\[
\leq \inf \{ (\tilde{k}^*_D(a, z)]^{q(F(\alpha))} : a \in G \} = d^\max_G(q \circ F, z), \quad z \in G.
\]

Step 3. If \( (d_G)_G \) satisfies (H), (M), and
\[
(E^-) \quad \prod_{\mu \in E} [m_E(\mu, \lambda)]^{p(\mu)} \leq d_E(p, \lambda), \quad (p, \lambda) \in \mathbb{R}_+^E \times E,
\]
then \( d^\min_G \leq d_G \) for any \( G \). The result remains true in the category of g.h.c.f. with integer-valued weights.

\( (26) \) Notice that in general \( d^\min_G(p, \cdot) \leq m_G(p, \cdot) \leq g_G(p, \cdot) \leq d^\max_G(p, \cdot) \) — cf. Examples 1.7.19, 1.7.20.
If in particular, then $m(a)$ properties of the generalized Möbius and Green functions (cf. [Jar-Jar-Pfl 2003]).

Directly from the Definitions 1.5.1 and 1.5.2 we get the following elementary properties of the generalized Möbius and Green functions (cf. [Jar-Jar-Pfl 2003]).

**Remark 1.6.1.** (a) $m_G(kp, \cdot) \geq [m_G(p, \cdot)]^k$, $k \in \mathbb{N}$ (27); $g_G(kp, \cdot) = [g_G(p, \cdot)]^k$, $k > 0$.

(b) If $p \leq q$, then $m_G(p, \cdot) \geq m_G(q, \cdot)$ and $g_G(p, \cdot) \geq g_G(q, \cdot)$, i.e. both systems $(m_G)G, (g_G)G$ satisfy condition (M) from Definition 1.5.3. In particular, if $A \subset B \subset G$, then $m_G(A, \cdot) \geq m_G(B, \cdot)$ and $g_G(A, \cdot) \geq g_G(B, \cdot)$.

(c) $m_G(p, \cdot)m_G(q, \cdot) \leq m_G(p + q, \cdot) \leq \min\{m_G(p, \cdot), m_G(q, \cdot)\}$, $g_G(p, \cdot)g_G(q, \cdot) \leq g_G(p + q, \cdot) \leq \min\{g_G(p, \cdot), g_G(q, \cdot)\}$.

In particular, $m_G(p, \cdot) \leq g_G(p, \cdot) \leq \inf_{a \in G} [g_G(a, \cdot)]^{p(a)} \leq \inf_{a \in G} [m_G(a, \cdot)]^{p(a)} = d_G^{\max}(p, \cdot)$.

If $|p|$ is finite, then $m_G(p, \cdot) \geq \prod_{a \in |p|} [m_G(a, \cdot)]^{p(a)}$, $g_G(p, \cdot) \geq \prod_{a \in |p|} [g_G(a, \cdot)]^{p(a)}$.

(27) Notice that in general $m_G(kp, \cdot) \neq [m_G(p, \cdot)]^k$; for instance, if $P \subset C$ is an annulus, then $m_{P}(k\chi_{(a)} \cdot) \neq [m_{P}(a \cdot)]^k$, $k \geq 2$; cf. [J-P 1993], Proposition 5.5.
1.6. Properties of the generalized M"obius and Green functions

(d) \( g_G(p, z) = \sup\{u(z) : u : G \twoheadrightarrow [0, 1], \log u \in PSH(G), u \leq \inf_{a \in G}[g_G(a, \cdot)]p(a)\}, \quad z \in G. \)

(e) Let \( G \subset \mathbb{C}^n, D \subset \mathbb{C}^m \) be domains and let \( F : G \twoheadrightarrow D \) be holomorphic. Then for any function \( q : D \twoheadrightarrow \mathbb{Z}^+ \) (resp. \( q : D \twoheadrightarrow \mathbb{R}^+ \)) we have
\[
m_D(q(F(z))) \leq m_G(q_F, z) \leq m_G(q \circ F, z),
g_D(q(F(z))) \leq g_G(q_F, z) \leq g_G(q \circ F, z), \quad z \in G,
\]

where
\[
a_F(a) := q(F(a)) \operatorname{ord}_a(F - F(a)), \quad a \in G. \tag{28}
\]

Thus both systems \((m_G)_G\) and \((g_G)_G\) satisfy condition (H) from Definition 1.5.3. In particular,
\[
m_D(B, F(z)) \leq m_G(F^{-1}(B), z), \quad g_D(B, F(z)) \leq g_G(F^{-1}(B), z), \quad B \subset D, \quad z \in G.
\]

(f) \( \log m_G(p, \cdot) \in \mathcal{C}(G) \cap PSH(G), \log g_G(p, \cdot) \in PSH(G) \) (we can argue as in the one-pole case; cf. [J-P 1993], Proposition 4.2.11, Lemma 4.2.3).

(g) If \( p \not\equiv 0 \), then for any \( z_0 \in G \) there exists an extremal function for \( m_G(p, z_0) \), i.e. a function \( f_{z_0} \in \mathcal{O}(G, E), \operatorname{ord}_z f_{z_0} \geq p(a), \ a \in G, \) and \( m_G(p, z_0) = \|f_{z_0}(z_0)\| \).

(h) If \( G_k \not\equiv G \) and \( p_k \not\equiv p \), then \( m_G(p_k, \cdot) \not\subset m_G(p, \cdot), \ g_G(p_k, \cdot) \not\subset g_G(p, \cdot) \).

Indeed, the case of the generalized M"obius function follows from a Monot argument (based on (g)).

In the case of the generalized Green function first recall that \( g_{G_k}(a, \cdot) \not\subset g_G(a, \cdot) \), \( a \in G; \) cf. [J-P 1993], Proposition 4.2.7(a). Let \( u_k := g_{G_k}(p_k, \cdot). \) Then \( \log u_k \in \mathcal{PSH}(G_k) \) (by (f)) and \( g_{G}(p, \cdot) \leq u_{k+1} \leq u_k \) on \( G_k \) (by (b) and (e)). Let \( u := \lim_{k \to +\infty} u_k. \) Obviously, \( u \geq g_G(p, \cdot) \) and \( \log u \in \mathcal{PSH}(G). \) Moreover, since \( u_k \leq [g_{G_k}(a, \cdot)]^{p_k(s)}, a \in G_k, \) we easily conclude that \( u \leq [g_G(a, \cdot)]^{p(s)}, a \in G. \) Hence, by (d), \( u = g_G(p, \cdot). \)

(i) Let \( P \subset G \) be a relatively closed pluripolar set such that \( p = 0 \) on \( P. \) Then \( g_G(p, \cdot) = g_G(p, \cdot) \) on \( G \setminus P \) (cf. [J-P 1993], Proposition 4.2.7(c)).

**Proposition 1.6.2** ([Jar-Jar-Pfl 2003]). \( g_G(p, \cdot) = \inf \{g_G(q, \cdot) : q \leq p, \#q < +\infty\}. \)

**Proof.** Let \( u := \inf \{g_G(q, \cdot) : q \leq p, \#q < +\infty\}. \) Obviously \( u \geq g_G(p, \cdot). \) To prove the opposite inequality we only need to show that \( \log u \) is plurisubharmonic. Observe that \( g_G(\max\{q_1, \ldots, q_N\}, \cdot) \leq \min\{g_G(q_1, \cdot), \ldots, g_G(q_N, \cdot)\} \). Thus we only need the following general result.

**Lemma 1.6.3.** Let \( (v_i)_{i \in A} \subset \mathcal{PSH}(\Omega) \) \( (\Omega \subset \mathbb{C}^n) \) be such that for any \( i_1, \ldots, i_N \in A \) there exists an \( i_0 \in A \) such that \( v_{i_0} \leq \min\{v_{i_1}, \ldots, v_{i_N}\} \). Then \( v := \inf_{i \in A} v_i \in \mathcal{PSH}(\Omega). \)

\(^{28}\) Observe that in the case where \( F \equiv \text{const} = b \) we have \( q_F \equiv +\infty \) if \( q(b) > 0 \) and \( q_F \equiv 0 \) if \( q(b) = 0 \).
Proof. It suffices to consider only the case \( n = 1 \). Take a disc \( \mathbb{B}(a, r) \Subset \Omega, \varepsilon > 0 \), and a continuous function \( w \in C(\partial \mathbb{B}(a, r)) \) such that \( w \geq v \) on \( \partial \mathbb{B}(a, r) \). We want to show that \( v(a) \leq \frac{1}{2\pi} \int_{0}^{2\pi} w(a + re^{i\theta})d\theta + \varepsilon \). For any point \( b \in \partial \mathbb{B}(a, r) \) there exists an \( i = i(b) \in A \) such that \( v_i(b) < w(b) + \varepsilon \). Hence there exists an open arc \( I = I(b) \subset \partial \mathbb{B}(a, r) \) with \( b \in I \) such that \( v_i(\lambda) < w(\lambda) + \varepsilon, \lambda \in I \). By a compactness argument, we find \( b_1, \ldots, b_N \in \partial \mathbb{B}(a, r) \) such that \( \partial \mathbb{B}(a, r) = \bigcup_{i=1}^{N} I(b_i) \). By assumption, there exists an \( i_0 \in A \) such that \( v_{i_0} \leq \min\{v_{i(b_1), \ldots, v_{i(b_N)}\} \). Then

\[
v(a) \leq v_{i_0}(a) \leq \frac{1}{2\pi} \int_{0}^{2\pi} v_{i_0}(a + re^{i\theta})d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} w(a + re^{i\theta})d\theta + \varepsilon. \quad \square \]
Proposition 1.6.6 ([Edi-Zwo 1998b], [Lär-Sig 1998b]). Let $G, D \subset \mathbb{C}^n$ be domains and let $F : G \rightarrow D$ be a proper holomorphic mapping.

(a) Let $q : D \rightarrow \mathbb{R}_+$. Assume that $\det F'(a) \neq 0$, $a \in F^{-1}(\{q\})$. Then
$$g_D(q, F(z)) = g_G(qF, z) = g_G(q \circ F, z), \quad z \in G.$$ 
In particular, if $B \subset D$ is such that $\det F'(a) \neq 0$, $a \in F^{-1}(B)$, then
$$g_D(B, F(z)) = g_G(F^{-1}(B), z), \quad z \in G.$$

(b) Assume that $D$ is convex. Then for any point $b \in D$ such that $\det F'(a) \neq 0$, $a \in F^{-1}(b)$, we have
$$m_D(b, F(z)) = m_G(F^{-1}(b), z), \quad z \in G.$$

Notice that (a) may be false if $\det F'(a) = 0$ for some $a \in F^{-1}(\{q\})$ — cf. Example 1.7.4. Moreover, (b) need not be true if $D$ is not convex — cf. Example 1.7.7.

For the behavior of the pluricomplex Green function under coverings see [Azu 1995], [Azu 1996].

Proof. (a) We only need to show $g_D(q, F(z)) \geq g_G(q \circ F, z), z \in G$; cf. Remark 1.6.1(e).

Put $S := \{z \in G : \det F'(z) = 0\}$, $\Sigma := F(S)$. It is well-known that
$$F|_{G \setminus F^{-1}(\Sigma)} : G \setminus F^{-1}(\Sigma) \rightarrow D \setminus \Sigma$$
is a holomorphic covering. Let $N$ denote its multiplicity.

Let $u : G \rightarrow [0, 1)$ be a logarithmically plurisubharmonic function such that
$$u(z) \leq C(a)\|z - a\|^{q(F(a))}, \quad a, z \in G.$$
Define
$$v(w) := \max\{u(z) : z \in F^{-1}(w)\}, \quad w \in D.$$ 
Since $F$ is proper, $\log v \in \mathcal{PSH}(D)$ (cf. [Kli 1991], Proposition 2.9.26). Take a $b \in D$ with $q(b) > 0$ (recall that $b \notin \Sigma$) and let $F^{-1}(b) = \{a_1, \ldots, a_N\}$ ($a_j \neq a_k$ for $j \neq k$).
There exist open neighborhoods $U_1, \ldots, U_N, V$ of $a_1, \ldots, a_N, b$, respectively, such that $F|_{U_j} : U_j \rightarrow V$ is biholomorphic, $j = 1, \ldots, N$. Let $g_j := (F|_{U_j})^{-1}$, $j = 1, \ldots, N$.
Shrinking the neighborhoods, if necessary, we may assume that there is a constant $M > 0$ such that $\|g_j(w) - a_j\| \leq M\|w - b\|$, $w \in V$. Then, for $w \in V$, we get
$$v(w) = \max\{u \circ g_j(w) : j = 1, \ldots, N\}$$
$$\leq \max\{C(a_j)\|g_j(w) - a_j\|^{q(b)} : j = 1, \ldots, N\}$$
$$\leq \max\{C(a_j) : j = 1, \ldots, N\}M^{q(b)}\|w - b\|^{q(b)}.$$
Consequently, $g_D(q, \cdot) \geq v$ and, therefore, $g_D(q, F(z)) \geq v(F(z)) \geq u(z), z \in G$, which gives the required inequality.

(b) By Remark 1.6.1(e) we only need to check the inequality “≥”. Since $D$ is convex, the Lempert theorem implies that $m_D(b, \cdot) = g_D(b, \cdot)$ (cf. [J-P 1993], Theorem 8.2.1). Hence, by (a) we get
$$m_D(b, F(z)) = g_D(b, F(z)) = g_G(F^{-1}(b), z) \geq m_G(F^{-1}(b), z), \quad z \in G. \quad \square$$
1.7. Examples

Example 1.7.1 ([Car-Ceg-Wik 1999]). Let
\[ T := \{(z_1, z_2) \in E_* \times E : |z_2| < |z_1|\} \]
be the Hartogs triangle. Let \( p : T \to \mathbb{R}_+ \). Consider the biholomorphism
\[ E_* \times E \ni (z_1, z_2) \mapsto (z_1, z_1 z_2) \in T. \]
The set \( E^2 \setminus (E_* \times E) \) is pluripolar. Hence, by Remark 1.6.1(e,i),
\[ g_T(p, F(z)) = g_{E_* \times E}(p \circ F, z) = g_{E^2}(p', z), \quad z \in E_* \times E, \]
where \( p' := p \circ F \) on \( E_* \times E \) and \( p' := 0 \) on \( \{0\} \times E \). In particular,
\[ g_T(a, z) = \max\{m_E(a_1, z_1), m_E(a_2/a_1, z_2/z_1)\}, \quad a = (a_1, a_2), \ z = (z_1, z_2) \in T. \]

Example 1.7.2. For any non-empty sets \( A_1, \ldots, A_n \subset E \) we have
\[
m_{E^n}(A_1 \times \cdots \times A_n, z) = g_{E^n}(A_1 \times \cdots \times A_n, z)
= \max\{m_E(A_1, z_1), \ldots, m_E(A_n, z_n)\}
= \max\left\{ \prod_{a_j \in A_j} m_E(a_j, z_j) : j = 1, \ldots, n \right\}, \quad z = (z_1, \ldots, z_n) \in E^n.
\]
In particular, for any non-empty set \( A \subset E \) we have
\[
m_{E^n}(A \times \{0\}^{n-1}, z) = g_{E^n}(A \times \{0\}^{n-1}, z) = \max\{m_E(A, z_1), |z_2|, \ldots, |z_n|\},
\quad z = (z_1, \ldots, z_n) \in E^n;
\]
cf. Example 1.7.17.

Indeed, by Propositions 1.6.2, 1.6.4 we may assume that \( A_1, \ldots, A_n \) are finite. Let
\[ F_j(\lambda) := \prod_{a \in A_j} \frac{\lambda - a}{1 - \overline{a} \lambda}, \quad \lambda \in E, \ j = 1, \ldots, n, \]
be the corresponding Blaschke products. The mapping
\[ E^n \ni (z_1, \ldots, z_n) \mapsto (F_1(z_1), \ldots, F_n(z_n)) \in E^n \]
is proper. Moreover, \( \det F'(z) = F_1'(z_1) \cdots F_n'(z_n) \neq 0 \) for \( z \in A_1 \times \cdots \times A_n \). Consequently, by Proposition 1.6.6,
\[
m_{E^n}(A_1 \times \cdots \times A_n, z) = g_{E^n}(A_1 \times \cdots \times A_n, z)
= g_{E^n}(0, F(z)) = \max\{|F_j(z)| : j = 1, \ldots, n\}
= \max\{m_E(A_1, z_1), \ldots, m_E(A_n, z_n)\}, \quad z = (z_1, \ldots, z_n) \in E^n.
\]

Example 1.7.3. Recall that for \( p = (p_1, \ldots, p_n) \in \mathbb{N}_0^n (n \geq 2) \), we put
\[ E_p := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\}. \]
Fix \((\nu_1, \ldots, \nu_n) \in \mathbb{N}^n\). The mapping
\[
\mathbb{B}_n \ni (z_1, \ldots, z_n) \xrightarrow{F} (z_1^{\nu_1}, \ldots, z_n^{\nu_n}) \in \mathbb{E}_{(1/\nu_1, \ldots, 1/\nu_n)}
\]
is proper. Let \((a_1, \ldots, a_n) \in \mathbb{B}_n\) be such that \(a_j^{\nu_j-1} \neq 0, j = 1, \ldots, n\), and let
\[
A := F^{-1}(F(a)) = \{(\varepsilon_1 a_1, \ldots, \varepsilon_n a_n) : \varepsilon_j \in \sqrt{1}, j = 1, \ldots, n\}.
\]
Then, by Proposition 1.6.6,
\[
g_{B^n}(A, z) = g_{\mathbb{S}^{n-1}}(F(a), F(z)), \quad z \in \mathbb{B}_n;
\]
roughly speaking, the multi-pole pluricomplex Green function for the Euclidean ball is expressed by the standard one-pole pluricomplex Green function for an ellipsoid.

Notice that for some special cases the function \(g_{\mathbb{S}^{n-1}}(F(a), F(\cdot))\) may be effectively calculated. For example, let \(n = 2, \nu_1 = 1, \nu_2 = 2, a = (0, s) \ (s \in (0, 1))\). Then \(A = \{(0, -s), (0, s)\}\) and
\[
g_{B^2}((0, -s), (0, s)), (z_1, z_2) = g_{\mathbb{S}^{1/2}}((0, s^2), (z_1, z_2^2)) = \begin{cases}
1 - \frac{(1-s^2)|z_1|^2 - |z_2|^2}{|1-sz_2|^2} & \text{if } |z_1| \geq |z_2 - s| \\
1 - \frac{(1-s^2)|z_1|^2 - |z_2|^2}{|1+sz_2|^2} & \text{if } |z_1| \geq |z_2 + s| \\
\frac{2(1-s^2 \text{Re} z_2^2)|z_1|^2 - s^2|z_1|^2 - z_2^2}{2|1-sz_2|^2} & \text{if } |z_1| < \min\{|z_2 - s|, |z_2 + s|\}
\end{cases},
\]
where
\[
\Delta := -4|z_1|^4(2 \text{Im} z_2^2)^2 + 4|z_1|^2(1 - s^2 \text{Re} z_2^2)|s^2 - s^2|z_1|^2 - z_2^2|^2 + |s^2 - s^2|z_1|^2 - z_2^2|^4;
\]
\cf \[\text{Edi-Zwo 1998b}\] (see also \[\text{Com 2000}\] for a different approach). We would like to point out that even \(\mathbb{S}^{1/2}\) for the case \(p = \{a_1, a_2\}, p(a_1) \neq p(a_2)\), a formula for \(g_{B^n}(p, \cdot)\) is not known \[\].

**Example 1.7.4.** Let \(\mathbb{B}_2 \ni (z_1, z_2) \xrightarrow{F} (z_1, z_2^2) \in \mathbb{E}_{(1,1/2)}, a := (0, 0)\). Then \(\det F'(0) = 0\) and \(g_{B^2}(0, \cdot) \neq g_{\mathbb{S}^{1/2}}(0, F(\cdot))\) for any \(s > 0\).

Indeed,
\[
g_{B^2}((0, 0), (z_1, z_2)) = h_{\mathbb{B}_2}(z_1, z_2) = \sqrt{|z_1|^2 + |z_2|^2},
\]
\[
g_{\mathbb{S}^{1/2}}((0, 0), (z_1, 0)) = h_{\mathbb{S}^{1/2}}(z_1, 0) = \frac{|z_2| + \sqrt{4|z_1|^2 + |z_2|^2}}{2},
\]
where \(h_D\) is the Minkowski function. In particular, for small \(t > 0\), we get
\[
g_{B^2}((0, 0), (t, t)) = t\sqrt{2}, \quad g_{\mathbb{S}^{1/2}}((0, 0), (t, t^2)) \approx t,
\]
which implies the required result.

**Example 1.7.5** ([Jaworowski 2003]). Let \(p = (p_1, \ldots, p_n) \in \mathbb{R}_{>0}^n, E := \mathbb{E}_p\). Put
\[
A = A_{E, k} := \{z \in E : z_1 \cdots z_k = 0\}, \quad k = 1, \ldots, n.
\]
Our aim is to find effective formulae for $m_E(A_{E,k}, z)$ and $g_E(A_{E,k}, z)$, where $z = (z_1, \ldots, z_n) \in \mathbb{E}$. It is clear that we may assume that

$$p_1 |z_1|^{2p_1} \leq \cdots \leq p_k |z_k|^{2p_k}. \quad (29)$$

Put

$$q_s := \sum_{j=1}^{k} \frac{1}{2p_j}, \quad r_s(z) := 1 - \sum_{j=k+1}^{n} |z_j|^{2p_j}, \quad c_s(z) := r_s(z)/q_s, \quad s = 1, \ldots, k \quad (r_n = 1),$$

$$d = d(z) := \max \left\{ s \in \{1, \ldots, k\} : 2p_s |z_s|^{2p_s} \leq c_s(z) \right\}. \quad (30)$$

$$R_E(A, z) := \prod_{j=1}^{d} \frac{|z_j|^{2p_j}}{c_d(z)} = \left( q_d^{d} \prod_{j=1}^{d} (2p_j)^{\frac{1}{2p_j}} \right) \frac{|z_1| \cdot \cdots \cdot |z_d|}{\left( 1 - \sum_{j=d+1}^{n} |z_j|^{2p_j} \right)^{\frac{1}{2p_d}}}. \quad \text{Then:}\$$

(a) $g_E(A, z) = R_E(A, z)$;

(b) $m_E(A, z) = g_E(A, z) = R_E(A, z)$ if $p_j \geq 1/2, j = d + 1, \ldots, n$;

(c) $m_E(A, z) = g_E(A, z) = R_E(A, z)$ for $k = 1, n = 2, p_2 \geq 1/2$;

(d) $m_E(A, z) \neq g_E(A, z)$ if there exists a $j_0 \in \{ k + 1, \ldots, n \}$ with $p_{j_0} < 1/2$, 

$|z_{j_0}| \neq 0$ small enough, 

$\ell = 1, \ldots, k, j_0, j = 0, \ell = k + 1, \ldots, j_0 - 1, j_0 + 1, \ldots, n$;

(e) $m_E(A, z) = g_E(A, z) = R_E(A, z)$ for $k = n = 2, p_1 \leq p_2$, and either $p_2 \geq 1/2$ or $8p_1 + 4p_2(1 - p_2) > 1$.

It is an open question whether $m_E(A, z) = g_E(A, z) = R_E(A, z)$ if $p_j \geq 1/2, j = k + 1, \ldots, n$ (with arbitrary $n$ and $k$).

Proof of (a). Step 1. $m_{E^n}(A_{E^n,k}, \zeta) = g_{E^n}(A_{E^n,k}, \zeta) = |\zeta_1 \cdots \zeta_k|, \zeta \in E^n$, where $A_{E^n,k} : = \{ \zeta \in E^n : \zeta_1 \cdots \zeta_k = 0 \}$.

Indeed, it is clear that $|\zeta_1 \cdots \zeta_k| \leq m_{E^n}(A_{E^n,k}, \zeta) \leq g_{E^n}(A_{E^n,k}, \zeta)$. It remains to prove that $u(\zeta) := g_{E^n}(A_{E^n,k}, \zeta) \leq |\zeta_1 \cdots \zeta_k|, \zeta \in E^n$. We proceed by induction on $k$ (with arbitrary $n$ and logarithmically plurisubharmonic function $u : E^n \rightarrow [0, 1]$) such that $u(\zeta) \leq C(a)||\zeta - a||, a \in A_{E^n,k}, \zeta \in E^n$.

For $k = 1$ the inequality follows from the Schwarz type lemma for logarithmically subharmonic functions $u(\zeta, \zeta_2, \ldots, \zeta_n)$.

For $k > 1$ we first apply the case $k = 1$ and get $u(\zeta_1, \ldots, \zeta_n) \leq |\zeta_1|, \zeta \in E^n$. Next we apply the inductive assumption to the functions $u(\zeta_1, \cdots, z_n)/|\zeta_1|, \zeta_1 \in E_{z_1}$.

Step 2. Consider the mapping

$$E^d \ni (\zeta_1, \ldots, \zeta_d) \mapsto \left( \frac{c_1(z)}{2p_1}, \ldots, \frac{c_d(z)}{2p_d}, z_{d+1}, \ldots, z_n \right) \in \mathbb{E}.$$
Step 3. $g_E(A, z) \geq R_E(A, z)$.

We may assume that $z_1 \ldots z_d \neq 0$.

First consider the case $d = k = n$. Put $f(\zeta) := q_n^{\alpha_n} \prod_{j=1}^{n} \zeta_j (2p_j)^{\frac{1}{2}}$, $\zeta \in \mathbb{E}$.

Observe that $|f(z)| = R_E(A, z)$ and $|f(\zeta)| \leq q_n^{\alpha_n} \left( \sum_{j=1}^{n} |\zeta_j|^{2p_j} / q_d \right) < 1$, $\zeta \in \mathbb{E}$ (31).

Thus $g_E(A, z) \geq m_E(A, z) \geq R_E(A, z)$.

Now assume that $d < n$. Put $E^r := \mathbb{E}_{(p_d+1, \ldots, p_n)}$. Observe that we only need to find a logarithmically plurisubharmonic function $v : E' \longrightarrow [0, 1)$, $v \neq 0$, such that

- $v(\zeta') \leq |\zeta_j|$, $\zeta' = (\zeta_{d+1}, \ldots, \zeta_n) \in E'$, $j = d + 1, \ldots, k$ (32),
- the mapping $E' \ni \zeta' \longrightarrow v(\zeta')r_d^{q_d}(\zeta') \in \mathbb{R}_{+}$ attains its maximum for $\zeta' = (z_{d+1}, \ldots, z_n)$ (33).

Indeed, suppose that such a $v$ is already constructed and let $M$ be the maximal value of the function $E' \ni \zeta' \longrightarrow v(\zeta')r_d^{q_d}(\zeta')$. Put

$$u(\zeta) := \frac{q_d}{M} \left( \prod_{j=1}^{d} |\zeta_j| (2p_j)^{\frac{1}{2}} \right) v(\zeta'), \quad \zeta = (\zeta_1, \ldots, \zeta_{d+n}) = (\zeta_1, \ldots, \zeta_d, \zeta').$$

Then $\log u \in \mathcal{P}SH(\mathbb{E})$ and $u(\zeta) \leq C(a)|\zeta_j| \leq C(a)\|\zeta - a\|$ for any $\zeta \in \mathbb{E}$ and $a \in A$ with $a_j = 0$, where $j \in \{1, \ldots, k\}$. Moreover, for $\zeta \in \mathbb{E}$ we have:

$$u(\zeta) \leq \frac{q_d}{M} \left( \sum_{j=1}^{d} |\zeta_j|^{2p_j} / q_d \right) q_d v(\zeta') = \frac{1}{M} \left( \sum_{j=1}^{d} |\zeta_j|^{2p_j} / \sum_{j=1}^{d} r_d(\zeta') \right) v(\zeta') r_d^{q_d}(\zeta') < 1.$$ 

Consequently, $u : \mathbb{E} \longrightarrow [0, 1)$ and, therefore,

$g_E(A, z) \geq u(z) = \frac{1}{M} R_E(A, z) v(\zeta')r_d^{q_d}(\zeta') = R_E(A, z)$.


We may assume that $z_{d+1}, \ldots, z_n \geq 0$. For $\alpha = (\alpha_{d+1}, \ldots, \alpha_n) \in \mathbb{R}_{+}^{n-d}$ define

$$v_\alpha(\zeta') := \left( \prod_{j=d+1}^{k} |\zeta_j|^{1+\alpha_j} \right) \left( \prod_{j=k+1}^{n} |\zeta_j|^{\alpha_j} \right).$$

Obviously $v : E' \longrightarrow [0, 1)$, $\log v \in \mathcal{P}SH(\mathbb{E}')$, and $v(\zeta') \leq |\zeta_j|$, $\zeta' \in E'$, $j = d + 1, \ldots, k$.

It is enough to find an $\alpha$ such that the function $E' \cap \mathbb{R}_{+}^{n-d} \ni \zeta' \longrightarrow v_\alpha(\zeta')r_d^{q_d}(\zeta')$ attains

---

(31) We have used the following elementary inequality

$$\prod_{j=1}^{d} a_j^{w_j} \leq \left( \sum_{j=1}^{d} w_j a_j \right) \left( \sum_{j=1}^{d} w_j \right), \quad a_1, \ldots, a_d \geq 0, \quad w_1, \ldots, w_d > 0.$$

(32) Notice that this condition is empty if $d = k$.

(33) $r_d(\zeta') = 1 - \sum_{j=d+1}^{n} |\zeta_j|^{2p_j}$. 

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its maximum at \( t' = z' \). In particular, \( \frac{\partial z_j}{\partial t_j}(z') = 0 \), \( j = d + 1, \ldots, n \). Hence

\[
0 = 1 + \alpha_j - 2p_jq_d \frac{z_j^{2p_j}}{r_d(z')} , \quad j = d + 1, \ldots, k ,
\]

\[
0 = \alpha_j - 2p_jq_d \frac{z_j^{2p_j}}{r_d(z')} , \quad j = k + 1, \ldots, n ,
\]

which gives formulas for \( \alpha_{d+1}, \ldots, \alpha_n \). To prove that there are no other points like this, rewrite the above equations in the form

\[
r_d(z') = \frac{2p_jq_d z_j^{2p_j}}{1 + \alpha_j} , \quad j = d + 1, \ldots, k ,
\]

\[
r_d(z') = \frac{2p_jq_d z_j^{2p_j}}{\alpha_j} , \quad j = k + 1, \ldots, n .
\]

The left side is decreasing in any of the variables \( z_{d+1}, \ldots, z_n \), while the right sides are increasing. Thus, at most one common zero is allowed.

It remains to check whether \( \alpha_j \geq 0 \), \( j = d + 1, \ldots, n \). Obviously, \( \alpha_j \geq 0 \), \( j = k + 1, \ldots, n \). In the remaining cases, using the definition of the number \( k \), we have:

\[
\alpha_j = \frac{2p_jq_d z_j^{2p_j} - r_d(z')}{r_d(z')} \geq 0 , \quad j = d + 1, \ldots, k .
\]

\[
\square
\]

**Proof of (b).** By the proof of (a), we only have to check whether \( m_{E}(A,z) \geq R_{E}(a,z) \) in the case where \( d < n \). First observe that it is sufficient to find a function \( h \in \mathcal{O}(E') \), \( h \neq 0 \), such that:

- \( h(\zeta') = 0 \) if \( \zeta_{d+1} \cdots \zeta_k = 0 \),
- the function \( E' \ni \zeta' \longrightarrow |h(\zeta')| r^{q_d}_d(\zeta') \in \mathbb{R}_+ \) attains its maximum for \( \zeta' = (z_{d+1}, \ldots, z_n) \).

Indeed, suppose that such an \( h \) is already constructed and let \( M \) be the maximal value of the function \( E' \ni \zeta' \longrightarrow |h(\zeta')| r^{q_d}_d(\zeta') \). Put

\[
f(\zeta) := \frac{q_d}{M} \left( \prod_{j=1}^{d} \zeta_j^{(2p_j) \frac{1}{q_d}} \right) h(\zeta') , \quad \zeta \in E .
\]

Obviously \( f(\zeta) = 0 \) for \( \zeta \in A \). Similarly as in (a) we prove that \( |f| < 1 \) and \( |f(z)| = R_{E}(A,z) \). Thus \( m_{E}(A,z) \geq |f(z)| = R_{E}(A,z) \).

To construct \( h \) assume that \( z_{d+1}, \ldots, z_n \geq 0 \) and define

\[
h_\alpha(\zeta') := \left( \prod_{j=d+1}^{k} \zeta_j^{\alpha_j} \right) \left( \prod_{j=k+1}^{n} e^{\alpha_j} \right) ,
\]

where \( \alpha = (\alpha_{d+1}, \ldots, \alpha_n) \in \mathbb{R}^{n-d}_+ \). It is enough to find an \( \alpha \) such that the function \( E' \cap \mathbb{R}^{n-d}_+ \ni t' \longrightarrow h_\alpha(t') r^{q_d}_d(t) \) attains its maximum at \( t' = (z_{d+1}, \ldots, z_n) \). Considering
the partial derivatives results in the following equations:

\[ 0 = \frac{1}{z_j} + \alpha_j - 2p_j q_d \frac{z_j^{2p_j-1}}{r_d(z')}, \quad j = d + 1, \ldots, k, \]

\[ 0 = \alpha_j - 2p_j q_d \frac{z_j^{2p_j-1}}{r_d(z')}, \quad j = k + 1, \ldots, n. \]

We continue as in the proof of (a). \qed

**Proof of (c).** Assertion (c) follows directly from (b). \qed

**Proof of (d).** Step 1. Suppose that \( m_{E}(A, z) = R_{E}(A, z) \). Let \( f \in O(E, E) \) be such that \( f|A \equiv 0 \) and \( |f(z)| = R_{E}(A, z) \) (cf. [Jar-Jar-Pfl 2003], Property 2.5). Put

\[ h(\zeta') := \frac{\partial^d f}{\partial z_1 \cdots \partial z_d}(0, \zeta'), \quad \zeta' \in E'. \]

We have \( h(\zeta') = 0 \) if \( \zeta_{d+1} \cdots \zeta_k = 0 \). For \( \zeta' \in E' \) consider the mapping

\[ E^d \ni (\xi_1, \ldots, \xi_d) \xrightarrow{\iota_{\zeta'}} (\xi_1 \left( \frac{c_d(\zeta')}{2p_1} \right)^{\frac{1}{2p_1}}, \ldots, \xi_d \left( \frac{c_d(\zeta')}{2p_d} \right)^{\frac{1}{2p_d}}, \zeta') \in E. \]

Applying the Schwarz lemma to the mapping \( f \circ \iota_{\zeta'} \), \( \zeta' \in E' \), we get

\[ |h(\zeta')| r_d^{\alpha_d}(\zeta') \leq q_d^{\alpha_d} \left( \prod_{j=1}^{d} (2p_j)^{\frac{1}{2p_j}} \right), \quad \zeta' \in E', \]

\[ |h(z')| r_d^{\alpha_d}(z') = q_d^{\alpha_d} \left( \prod_{j=1}^{d} (2p_j)^{\frac{1}{2p_j}} \right). \]

Thus we have constructed a mapping \( h \) as in the proof of (b) (consequently, the equality \( m_{E}(A, z) = R_{E}(A, z) \) is equivalent to the existence of the mapping \( h \)).

**Step 2.** For any \( p \in (0, 1) \) and \( q > 0 \) there exists \( c = c(p, q) \in (0, 1) \) such that for any function \( f \in O(E) \) if the function \( E \ni \lambda \longrightarrow |f(\lambda)|(1 - |\lambda|^p)^q \) attains its maximum at \( \lambda_0 \neq 0 \), then \( |\lambda_0| \geq c \).

Let

\[ \varphi(t) := \frac{1}{(1 - tp)^q}, \quad t \in [0, 1). \]

Observe that there exists a \( b \in (0, 1) \) such that \( \varphi \) is strictly concave on \([0, b)\). Moreover,

\[ \lim_{t \to 0^+} \frac{\varphi(t) - \varphi(0)}{t} = +\infty. \]

Consequently, there exists a \( c \in (0, b) \) such that

\[ \varphi(0) + \frac{b}{c}(\varphi(c) - \varphi(0)) > \varphi(b) + 2. \]
Suppose that \( f \in \mathcal{O}(E) \) is such that the function \( E \ni \lambda \mapsto |f(\lambda)|/|\psi(\lambda)| \) attains its maximum at \( \lambda_0 \neq 0 \) with \( |\lambda_0| < c \). We may assume that \( |f(\lambda_0)| = \psi(\lambda_0)|. \) Consider the function
\[
[0, b] \ni t \mapsto |f(0)| + t|f(\lambda_0) - f(0)|.
\]
From \( \psi(0) = |f(0)| \leq \psi(0) = 1, \psi(|\lambda_0|) \geq \psi(|\lambda_0|) \), and the convexity condition we get:
\[
\psi(b) = |f(0)| + \frac{b}{|\lambda_0|}|f(\lambda_0) - f(0)| \geq \psi(0) + \frac{b}{|\lambda_0|}|\psi(|\lambda_0|) - \psi(0)|
\]
\[
\geq \varphi(0) + \frac{b}{c}|\psi(\varphi) - \psi(0)| > \varphi(b) + 2.
\]
The Schwarz lemma and the maximum principle imply that there exists a \( \lambda_0 \in E \) with \( |\lambda_0| = b \) and
\[
|f(\lambda_0)| - |f(0)| \geq \frac{|f(\lambda_0) - f(0)|}{|\lambda_0|}.
\]
This means that
\[
|f(\lambda_0)| \geq |f(\lambda_0) - f(0)| - |f(0)| = |f(0)| + |f(\lambda_0) - f(0)| - 2|f(0)|
\]
\[
\geq \psi(b) - 2|f(0)| > \varphi(b) + 2 - 2|f(0)| \geq \varphi(b) = \varphi(\lambda_0);
\]
contradiction.

**Step 3.** We may assume that \( p_{k+1} < 1/2 \). Assume that \( 0 < |z_j| < \varepsilon, j = 1, \ldots, k+1, \)
\( z_j = 0, j = k+2, \ldots, n, \) with \( 0 < \varepsilon < c(2p_{k+1}, q_k) \). Observe that \( d(z) = k \) provided \( \varepsilon \) is small enough. Let \( h \) be as in Step 1. Then the mapping
\[
E \ni \lambda \mapsto [h(\lambda, 0, \ldots, 0)](1 - |\lambda|^{2p_{k+1}})^{q_k}
\]
attains its maximum at \( \lambda = z_{k+1} \), which contradicts Step 2. \( \square \)

**Proof of (e).** See [JarW 2003]. \( \square \)

**Example 1.7.6.** Let \( P = P(R) := \{ z \in \mathbb{C} : 1/R < |z| < R \} \) \( (R > 1) \). Put \( q := 1/R^2 \) and let
\[
\Pi(a, z) = \Pi_R(a, z) := \prod_{\nu=1}^{\infty} \frac{(1 - a2^\nu)(1 - z2^\nu)}{(1 - a2^\nu)(1 - z2^\nu - 1)},
\]
\[f(a, z) = f_R(a, z) := \left(1 - \frac{z}{a}\right)\Pi(a, z), \quad 1/R < a < R, \ z \in P.
\]
Using the same methods as in the proof of Proposition 5.5 in [J-P 1993], one can prove that for any function \( p : P \to \mathbb{Z}_+ \) such that \( |p| = \{a_1, \ldots, a_N\} \) is finite, if \( a_j = |a_j|e^{i\phi_j}, |a_j| = R^{1 - 2s_j}, s_j \in (0, 1), j = 1, \ldots, N, \) then we get
\[
m_p(p, z) = \frac{f(b_0, |z|)}{R^\ell} \prod_{j=1}^N |f(|a_j|, e^{-i\phi_j}z)|^{k_j}, \quad z \in P,
\]
where
- \( \ell = \ell(p) := [s_1 + \cdots + s_N], \)
- \( b = b(p) := R^{1 - 2(\ell - (s_1 + \cdots + s_N))}, \)
1.7. Examples

- \( f(R, \cdot) \equiv 1 \).

**Example 1.7.7.** If \( D \) is not convex, then Proposition 1.6.6(b) need not be true.

Indeed, let \( P(R), \Pi_R \), and \( f_R \) be as Example 1.7.6. Consider \( F : P(R) \rightarrow P(R^2) \), \( F(z) := z^2 \), and suppose that \( m_{P(R)}(1, z^2) = m_{P(R)}(\{ -1, +1 \}, z) \), \( z \in P(R) \), \( R > 1 \). Then, using Example 1.7.6, we get

\[
\frac{f_R(z, 1 - |z|^2)}{f_R(z)} = \frac{1}{R(z)} |f_R(z)|, \quad z \in P(R).
\]

Consequently,

\[
\frac{1}{R(z)} (1 + |z|^2) \Pi_{R^2}(1, -|z|^2)(1 - z^2) \Pi_{R^2}(1, z^2) = |1 - z| \Pi_R(1, z)(1 + z) \Pi_R(1, -z),
\]

and hence

\[
\frac{1}{R(z)} (1 + |z|^2) \Pi_{R^2}(1, -|z|^2)(1 - z^2) \Pi_{R^2}(1, z^2) = \Pi_R(1, z) \Pi_R(1, -z), \quad z \in P(R);
\]

contradiction (at least for big \( R \) (take \( z = 1 \) and then let \( R \rightarrow +\infty \)).

**Remark 1.7.8.** Let \( G \subset C^\infty \) and assume that \( p : G \rightarrow \mathbb{R}_+ \), \( |p| = \{ a_1, \ldots, a_N \} \). Directly from the definition of the function \( d_{G, r}^\min \) we get the following estimate:

\[
d_{G, r}^\min (p, z) = \sup \left\{ \prod_{j=1}^s [m_E(\mu_j, z)]^{\max p(B_j)} : s \in \mathbb{N}, \mu_1, \ldots, \mu_s \in E, \mu_j \neq \mu_k (j \neq k), B_1 \cup \cdots \cup B_s = |p|, B_j \cap B_k = \emptyset (j \neq k), \exists f_{j \in \mathcal{O}(G, E)} : f|B_j \equiv \mu_j, j = 1, \ldots, s \right\}
\]

\[
\leq \sup \left\{ \prod_{j=1}^s [m_G(B_j, z)]^{\max p(B_j)} : s \in \mathbb{N}, B_1 \cup \cdots \cup B_s = |p|, B_j \cap B_k = \emptyset (j \neq k) \right\}
\]

\[
= d_G^\min (p, z), \quad z \in G.
\]

Recall that in the case where \( p = \chi_A \) we have \( d_{G, r}^\min (\chi_A, \cdot) \equiv m_G(A, \cdot) \equiv d_G^\min (\chi_A, \cdot) \) (cf. Proposition 1.5.4).

In particular, if \( N = 2, |p| = \{ a, b \}, \alpha = p(a) \geq p(b) = \beta \), then we get

\[
d_{G, r}^\min (\alpha \chi_{\{ a \}} + \beta \chi_{\{ b \}}, z) \leq \max \left\{ [m_G(a, z)]^{\alpha} [m_G(b, z)]^{\beta}, [m_G(\{ a, b \}, z)]^{\alpha} \right\}
\]

\[
= d_G^\min (\alpha \chi_{\{ a \}} + \beta \chi_{\{ b \}}, z), \quad z \in G.
\]

Notice that in general \( d_{G, r}^\min (\alpha \chi_{\{ a \}} + \beta \chi_{\{ b \}}, \cdot) \neq d_G^\min (\alpha \chi_{\{ a \}} + \beta \chi_{\{ b \}}, \cdot) \).

In fact, let \( G = P \) be an annulus as in Example 1.7.6. Take \( 1/R < a, b < R, a \neq b, ab \neq 1 \), \( p := 2\chi_{\{ a \}} + \chi_{\{ b \}} \). We are going to show that there exists a \( z \in P \) such that

\[
m_G(p, z)^2 = [m_G(\{ a, b \}, z)]^2 > m_G(\{ a, b \}, z),
\]

\[
d_{G, r}^\min (p, z) < [m_G(\{ a, b \}, z)]^2 m_G(b, z)
\]

\[
= \max \left\{ [m_G(\{ a, b \}, z)]^2 m_G(b, z), [m_G(\{ a, b \}, z)]^2 \right\} = d_G^\min (p, z).
\]
First observe that there are points $z$ (near $b$) such that
$$[m_P(a, z)]^2 m_P(b, z) > [m_P(a, b, z)]^2.$$ For, using the effective formula from Example 1.7.6, we have:
$$[m_P(a, z)]^2 m_P(b, z) = \left( \frac{f(1/a, -|z|)}{R|z|} |f(a, z)| \right)^2 \frac{f(1/b, -|z|)}{R|z|} |f(b, z)|, $$
$$[m_P(a, b, z)]^2 = \left( \frac{f(c, -|z|)}{(R|z|)^2} |f(a, z) f(b, z)| \right)^2,$$
with $c = c(a, b)$ and $\ell = \ell(a, b)$ as in Example 1.7.6. Consequently, we only need to find
$a \in P \setminus \{a, b\}$ such that
$$\left( \frac{f(1/a, -|z|)}{R|z|} \right)^2 \frac{f(1/b, -|z|)}{R|z|} > \left( \frac{f(c, -|z|)}{(R|z|)^2} \right)^2 |f(b, z)|.$$ Observe that at $z = b$ the right hand side of the above formula is zero while the left hand
side is strictly positive. Thus, by continuity, we can easily find the required $z$, say $z_0$.

Let $\varphi \in \mathcal{O}(F, E)$ be an extremal function for $d_p^{\min}(p, z_0)$ (Proposition 1.8.4). We may
assume that $\varphi(a) = 0$. There are two cases:
(a) $\varphi(b) = 0$: Then $d_p^{\min}(p, z_0) = |\varphi(z_0)|^2 \leq [m_P\{a, b, z_0\}]^2 < d_p(p, z_0)$.
(b) $\varphi(b) \neq 0$: Then $d_p^{\min}(p, z_0) = |\varphi(z_0)|^2 m_E(\varphi(b), \varphi(z_0)) \leq [m_P(a, z_0)]^2 m_P(b, z_0) = d_p(p, z_0)$. The equality $d_p^{\min}(p, z_0) = d_p(p, z_0)$ would imply that $\varphi$ is simultaneously
extremal for $m_P(a, z_0)$ and $m_P(b, z_0)$. Using Robinson’s lemma ([J-P 1993], Lemma 5.6),
we know that such extremal functions are uniquely determined up to rotations. Hence
$$\varphi(z) = e^{i\theta} \frac{f(1/a, -|z|)}{Rz} f(\alpha, z) = e^{i\theta} \frac{f(1/b, -|z|)}{Rz} f(b, z);$$
contradiction (both sides have different zeros).

**Example 1.7.9 ([Jar-Pfl 1999a]).** Let $F$ be a primitive polynomial of $n$–complex variables i.e. $F \in \mathcal{P}(\mathbb{C}^n)$ is a polynomial which cannot be represented in the form $F = f \circ Q$,
where $f$ is a polynomial of one complex variable of degree $\geq 2$ and $Q \in \mathcal{P}(\mathbb{C}^n)$. Notice
that a monomial $z_\alpha^\alpha$ $(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n)$ is primitive iff the numbers $\alpha_1, \ldots, \alpha_n$
are relatively prime.

One can prove (cf. [Cyg 1992]) that there exists a finite set $S \subset \mathbb{C}$ such that for any
$b \in \mathbb{C} \setminus S$:
- the fiber $F^{-1}(b)$ is connected,
- $F'(a) \neq 0$ for $a \in F^{-1}(b)$. Thus, if $b \notin S$, then the fiber $F^{-1}(b)$ is a connected $(n - 1)$–dimensional
algebraic manifold. In particular, for any $b \notin S$, the fiber $F^{-1}(b)$ has the plurisubharmonic Liouville property, i.e. any plurisubharmonic function $u : F^{-1}(b) \rightarrow [-\infty, 0]$ is constant
(cf. [Jar-Pfl 1999a], Proposition 6).

Put $r(a) := \text{ord}_a(F - F(a))$. Let $D \subset \mathbb{C}$ be a domain. Put $G := F^{-1}(D)$. Then
$G$ is a domain. Indeed, since the set $F^{-1}(S)$ is thin, it suffices to prove that the set
$G_0 := F^{-1}(D \setminus S)$ is connected. Suppose $G_0 = U_1 \cup U_2$, where $U_1, U_2$ are non-empty
disjoint open sets. Put $B_j := \{w \in D \setminus S : F^{-1}(w) \subset U_j\}, j = 1, 2$. Since the fibers over
points from $D \setminus S$ are connected, we conclude that $B_1, B_2$ are disjoint and $B_j = F(U_j)$, $j = 1, 2$. In particular, $B_j$ is open, non-empty, and $D \setminus S = B_1 \cup B_2$; contradiction.

(a) Let $p : G \longrightarrow \mathbb{R}^+$ be such that $|p|$ is finite. Then

$$g_G(p, z) = g_D(p^F, F(z)), \quad z \in G,$$

(1.7.7)

where

$$p^F(b) := \max \left\{ \frac{p(a)}{r(a)} : a \in |p| \cap F^{-1}(b) \right\}, \quad b \in D.$$ 

In particular,

$$g_G(a, z) = [g_D(F(a), F(z))]^{1/r(a)}, \quad a, z \in G.$$ 

(b) Let $p : G \longrightarrow \mathbb{Z}^+$ be such that $|p|$ is finite. Then

$$m_G(p, z) = m_D(p^F, F(z)), \quad z \in G,$$

(1.7.8)

where

$$p^F(b) := \max \left\{ \frac{\lceil p(a) \rceil}{\lceil r(a) \rceil} : a \in |p| \cap F^{-1}(b) \right\}, \quad b \in D.$$ 

In particular,

$$m_G(k\chi_{(a)}, z) = m_D(k'\chi_{(F(a))}, F(z)), \quad a, z \in G, \quad k \in \mathbb{N},$$

where $k' := \left\lceil \frac{k}{r(a)} \right\rceil$.

Indeed, in both cases the inequalities “$\geq$” follow from Remark 1.6.1(e).

To prove the opposite inequality in (a), take a logarithmically plurisubharmonic function $u : G \longrightarrow [0, 1)$ such that

$$u(z) \leq C(a)\|z - a\|^{p(a)}, \quad a, z \in G.$$ 

For any $b \in D \setminus S$, the function $u|_{F^{-1}(b)}$ is constant. Consequently, there exists a logarithmically subharmonic function $\tilde{u} : D \setminus S \longrightarrow [0, 1)$ such that $u = \tilde{u} \circ F$ on $G \setminus F^{-1}(S)$. The function $\tilde{u}$ extends to a logarithmically subharmonic function on $D$ (the extended function will be denoted by the same letter $\tilde{u}$). By the identity principle for plurisubharmonic functions we get $u = \tilde{u} \circ F$ in $G$. We want to show that

$$\tilde{u}(w) \leq \text{const}(b)\|w - b\|^{p^F(b)}, \quad b, w \in D;$$

(1.7.9)

then $\tilde{u} \leq g_D(p^F, \cdot)$ and hence $u = \tilde{u} \circ F \leq g_D(p^F, F)$, which gives the required inequality.

Fix a $b \in D$ with $p^F(b) > 0$, and let $a \in F^{-1}(b)$ be such that $p(a) > 0$. Observe that there exist $\varepsilon > 0$, $\delta > 0$, and $M > 0$ such that $\mathbb{B}(b, \varepsilon) \subset D$ and

$$\forall_{w \in \mathbb{B}(b, \varepsilon)} \exists_{z(w) \in \mathbb{B}(a, \delta)} : F(z(w)) = w, \quad \|z(w) - a\|^{p^F(b)} \leq M|w - b|.$$ 

Indeed, let $X \in \mathbb{C}^n$, $\|X\| = 1$, be such that $\text{ord}_{0}(\varphi - b) = r(a)$, where $\varphi(\lambda) := F(a + \lambda X)$. Then $|\varphi(\lambda) - b| \geq (1/M)|\lambda|^{r(a)}$ for $|\lambda| < \delta$ (where $M > 0$, $\delta > 0$) and $\varphi$ is open. Consequently, $\varphi(\mathbb{B}(\delta)) \supset \mathbb{B}(b, \varepsilon)$ for some $\varepsilon > 0$. Thus for any $w \in \mathbb{B}(b, \varepsilon)$ there exists a $\lambda(w) \in \mathbb{B}(\delta)$ such that $z(w) := a + \lambda(w)X$ satisfies all the required conditions.
Example 1.7.10. Let \( \alpha \in \mathbb{N}^n \) and \( D = E \) be such that \( \text{rank} A = m \), where \( A := [\alpha_{j,k}] \). Assume that \( \alpha_{j,1}, \ldots, \alpha_{j,n} \in \mathbb{Z}^n \), \( j = 1, \ldots, m \leq n - 1, m \geq 2 \), be such that rank \( A = m \), where \( A := [\alpha_{j,k}] \). Assume that \( AZ^m = Z^m \) \(^{(35)} \). Let \( F = (F_1, \ldots, F_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m \)

\(^{(34)} \) Observe that \( r(a) = 1 \) if \( a_1 \ldots a_n \neq 0 \), and \( r(a) = \text{the sum of those } \alpha_i \text{ for which } \alpha_i = 0 \) if \( a_1 \ldots a_n = 0 \).

\(^{(35)} \) One can prove that \( AZ^m = Z^m \) iff the greatest common divisor of all determinants of \( m \times m \) submatrices of \( A \) equals 1.
be given by the formula $F_j(z) := z^{\alpha_j}$, $j = 1, \ldots, m$. Define

$$G := F^{-1}(E^m) = \{ z \in \mathbb{C}^n : |z^{\alpha_j}| < 1, j = 1, \ldots, m \}.$$ 

Let $p : G \rightarrow \mathbb{R}_+ \ (\text{resp.} \ p : G \rightarrow \mathbb{Z}_+)$ be such that $|p|$ is finite and for any $a \in |p|$ we have rank $F'(a) = m$ (in particular, $r(a) = 1$). Then

$$g_G(p, z) = g_{E^m}(p^F, F(z)), \quad m_G(p, z) = m_{E^m}(p^F, F(z)) \quad z \in G, \quad (1.7.10)$$

where

$$p^F(b) := \max \{ p(a) : a \in |p| \cap F^{-1}(b) \}, \quad b \in E^m.$$

In particular, if $p = \chi_{(a)}$, then

$$g_G(a, z) = g_{E^m}(F(a), F(z)), \quad m_G(a, z) = m_{E^m}(F(a), F(z)), \quad z \in G.$$

Indeed, put $V_0 := \{ w = (w_1, \ldots, w_m) \in \mathbb{C}^m : w_1 \cdots w_m = 0 \}$ and observe that for any $w \in E^m \setminus V_0$ the fiber $V_w := F^{-1}(w)$ is connected (37).

For (the proof is due to W. Zwonek), let $w = (u_1 e^{2\pi i \theta_1}, \ldots, u_m e^{2\pi i \theta_m})$. Take arbitrary two points $a, b \in F^{-1}(w) = (r_1 e^{2\pi i \varphi_1}, \ldots, r_m e^{2\pi i \varphi_m})$, $b = (s_1 e^{2\pi i \psi_1}, \ldots, s_m e^{2\pi i \psi_m})$. We have $F(r) = F(s) = u$, $A \varphi = \theta \mod \mathbb{Z}^n$, $A \psi = \theta \mod \mathbb{Z}^n$. We have to find a curve $\gamma : [0, 1] \rightarrow F^{-1}(w)$ such that $\gamma(0) = a$, $\gamma(1) = b$. Write

$$\gamma(t) = (R_1(t)e^{2\pi i (\varphi_1 + \sigma_1(t))}, \ldots, R_m(t)e^{2\pi i (\varphi_m + \sigma_m(t))}),$$

where $R : [0, 1] \rightarrow \mathbb{R}^n$ is continuous, $\sigma : [0, 1] \rightarrow \mathbb{R}^n$ is such that the mapping $t \rightarrow (\varsigma_1 + \sigma_1(t)), \ldots, \varsigma_m + \sigma_m(t))$ is continuous, $F(R(t)) = u$, $A \sigma(t) = 0 \mod \mathbb{Z}^m$, $t \in [0, 1]$, $R(0) = r$, $R(1) = s$, $\sigma(0) = \sigma(1) = \psi - \varphi \mod \mathbb{Z}^n$.

Note that the set $\{ x \in \mathbb{R}^n : F(x) = u \}$ is connected. Hence we can easily find an $R$ with the required properties. To find a $\sigma$ it would be sufficient to know that the set $T := \{ x \in \mathbb{R}^n : A x \mod \mathbb{Z}^n \}$ is connected. Since $AZ^n = \mathbb{Z}^n$, we get $T = \{ x \in \mathbb{R}^n : A x \mod \mathbb{Z}^n \}$, which directly implies that $T$ is connected (because $A^{-1}(0)$ is connected).

The inequalities "$>$" in (1.7.10) follow from Remark 1.6.1(c).

For the proof of "$<$" in the case of a generalized Green function, let $u : G \rightarrow [0, 1]$ be such that $\log u \in \mathcal{PSH}(G)$ and $u(z) \leq C(a) \|z - a\|^{p(a)}$ for any $a, z \in G$. For any $w \in E^m \setminus V_0$, since $V_w$ is connected algebraic set, the function $u|_{V_w}$ is constant. Hence there exists a logarithmically plurisubharmonic function $v : E^m \setminus V_0 \rightarrow [0, 1]$ such that $u = v \circ F$ on $G \setminus F^{-1}(V_0)$. By the Riemann type extension theorem for plurisubharmonic functions, $v$ extends to a logarithmically plurisubharmonic function on $E^m$. By the identity principle for plurisubharmonic functions we get $u = v \circ F$ in $G$.

Fix a $b \in E^m$ with $p^F(b) > 0$ and let $a \in F^{-1}(b)$ be such that $p(a) > 0$. By our assumption (rank $F'(a) = m$) there exists an $m$–dimensional vector subspace $L \subset \mathbb{C}^n$ such that the mapping $L \ni z \rightarrow F(a + z)$ is biholomorphic in a neighborhood of $0 \in L$. Then $\|g(z) - b\| \geq (1/M) \|z\|$, $z \in L \cap \mathbb{B}((\delta))$ (for some $M, \delta > 0$) and $g(L \cap \mathbb{B}(\delta)) \supset \mathbb{B}(b, \varepsilon)$

\[\text{(36)}\]

Notice that the formula (1.7.10) may be not true if rank $F'(a) \leq m - 1$ — cf. Example 1.7.13.

\[\text{(37)}\]

In fact, one can prove that $V_w$ is connected iff $AZ^n = \mathbb{Z}^n$.
To show that the inequality (cf. Example 1.7.9). Hence, for any \( p \leq 1 \) Holomorphically invariant objects (even in the case where \( \{a|\bar{1}, \cdots, m\}, s \geq 1 \) it suffices \( \left( \begin{array}{c} \beta_1 \\ \vdots \\ \beta_m \end{array} \right) \) Note that (ii) gives an effective formula for \( g_G \). To prove that \( L \leq R \) it suffices to show that \( L(z) \geq R(z) \) for any \( z \in G_0 := G \cap ((\mathbb{C}_+)^n \times \mathbb{C}^{n-\delta}) \).

\( ^{38} \) We do not assume that \( m \leq n - 1 \), rank \( A = m \), \( AZ^n = Z^m \).

\( ^{39} \) Note that (ii) gives an effective formula for \( g_G(a, c) \).
By virtue of (i), for any \( k = 1, \ldots, s \), the system of equations
\[
\alpha_{j,s+1}x_{s+1} + \cdots + \alpha_{j,n}x_n = -\alpha_{j,k}, \quad j = 1, \ldots, m,
\]
has a rational solution \((Q_{s+1,k}/\mu_k, \ldots, Q_{n,k}/\mu_k)\) with \(Q_{s+1,k}, \ldots, Q_{n,k} \in \mathbb{Z}, \mu_k \in \mathbb{N}\). Put \(Q_{k,k} := \mu_k\) and \(Q_{j,k} := 0\), \(j, k = 1, \ldots, s, j \neq k\). Then
\[
\alpha_{j,k}Q_{1,k} + \cdots + \alpha_{j,n}Q_{n,k} = 0, \quad j = 1, \ldots, m, \quad k = 1, \ldots, s. \tag{1.7.11}
\]

Let \(Q_j := (Q_j, i, \ldots, Q_j, i) \in \mathbb{Z}^s, j = 1, \ldots, n\). Define \(\Phi : (\mathbb{C}_*)^s \times \mathbb{C}^{n-s} \rightarrow (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}\),
\[
\Phi(\xi, \eta) := (\xi^1_1, \ldots, \xi^1_s, \xi^{s+1}_1, \ldots, \xi^{s+1}_n, \xi^{s+2}_1, \ldots, \xi^{s+2}_n, \ldots, \xi^{n-s}_1, \ldots, \xi^{n-s}_n) = (\xi^1_1, \ldots, \xi^1_s, \xi^{s+1}_1, \ldots, \xi^{s+1}_n, \xi^{s+2}_1, \ldots, \xi^{s+2}_n, \ldots, \xi^{n-s}_1, \ldots, \xi^{n-s}_n),
\]
\(\xi, \eta = (\xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_n) \in (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}\).

Observe that \(\Phi\) is surjective. Indeed, for \(z = (z_1, \ldots, z_n) \in (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}\), take an arbitrary \(\xi_j \in (z_j)^{1/\mu_j}, j = 1, \ldots, s\), and define \(\eta_j := z_{s+j}/\xi^{Q_{s+j}}, j = 1, \ldots, n-s\).

Moreover, if \(z = \Phi(\xi, \eta)\), then by (1.7.11) we get
\[
z^{\alpha_j} = c^{\alpha_j}_1\xi^{s+1}_1 \cdots \alpha_j \cdots \xi^{n-s}_n = \eta^{\beta_j}, \quad j = 1, \ldots, s. \tag{1.7.12}
\]

Let \(D := \{\eta \in \mathbb{C}^{n-s} : |\eta^{\beta_j}| < 1,\ j = 1, \ldots, m\}\). Using (1.7.12) we get the equality
\[
\Phi((\mathbb{C}_*)^s \times D) = \mathbb{G}_0.
\]

Fix a \(\xi_0 \in (\mathbb{C}_*)^s\) such that \(a = \Phi(\xi_0, 0)\). Then, for any \(z = \Phi(\xi, \eta) \in \mathbb{G}_0\), we have
\[
g_c(a, z) = g_c(\Phi(\xi_0, 0), \Phi(\xi, \eta)) = g_{(\mathbb{C}_*)^s \times D}(\xi_0, 0, (\xi, \eta)) = g_D(0, \eta) = h_D(\eta)
\]
\[
= \max\{|\eta^{\beta_j}|^{1/r_j} : j = 1, \ldots, m\} = \max\{|z^{\alpha_j}|^{1/r_j} : j = 1, \ldots, m\}.
\]

The implications (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv) \(\Rightarrow\) (v) are trivial.

(v) \(\Rightarrow\) (i). Suppose that \(\text{rank} \ A < \text{rank} \ A\). We may assume that
\[
2 \leq t := \text{rank} \ A = \text{rank} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{bmatrix}, \quad \text{rank} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_t \end{bmatrix} < t.
\]

Then there exist \(c_1, \ldots, c_t \in \mathbb{Z}\) such that \(c_1 \beta_1 + \cdots + c_t \beta_t = 0\) and \(|c_1| + \cdots + |c_t| > 0\).

To simplify notation, assume that \(c_1, \ldots, c_u \geq 0, c_{u+1}, \ldots, c_t < 0\) for some \(1 \leq u \leq t-1\). Let
\[
d := a^{c_1 \alpha_1 + \cdots + c_t \alpha_t}, \quad r := c_1 r_1 + \cdots + c_u r_u = -(c_{u+1} r_{u+1} + \cdots + c_t r_t),
\]
\[
f(z) := \frac{z^{c_1 \alpha_1 + \cdots + c_u \alpha_u} - d z^{-c_1 \alpha_1 + \cdots + c_t \alpha_t}}{1 + |d|}, \quad z \in \mathbb{G}.
\]

Observe that \(f \in \mathcal{O}(G, E), \\text{ord}_a f \geq r+1, \text{ and } f \neq 0\) (because \(\alpha_1, \ldots, \alpha_t\) are linearly independent). Fix a \(b = (b', b'') \in G \subset \mathbb{C}^s \times \mathbb{C}^{n-s}\) with \(f(b) \neq 0\). Observe that \(f(b', \theta b'') = \theta^r f(b), 0 \leq \theta \leq 1\). Thus we get
\[
\frac{1}{\theta} m^{(r+1)}(a, (b', \theta b'')) \geq \frac{1}{\theta} |f(b', \theta b'')|^{1/(r+1)} = \theta^{-1/(r+1)} |f(b)|^{1/(r+1)} \rightarrow +\infty;\]

contradiction.
Moreover, one can prove (see (c)) that

\[ m_G(0, z) = g_G(z) = \max\{|z_1 z_2|^{1/2}, |z_1 z_3|^{1/2}\}, \quad z \in G. \]

Moreover, one can prove (see (c)) that

\[ m_G^{(2p)}(0, z) = g_G(0, z) = \max\{|z_1 z_2|^{1/2}, |z_1 z_3|^{1/2}\}, \quad p \in \mathbb{N}, \]
\[ m_G^{(2p+1)}(0, z) = \max\{|z_1 z_2|^{\frac{p+1}{2p+1}}, |z_1 z_3|^{\frac{p+1}{2p+1}}\}, \quad p \in \mathbb{Z}_+, \quad z \in G. \]

(a) It is well known that

\[ \alpha + \beta \neq 0, \quad \beta \neq 0. \]

(b) By Example 1.7.11, if \( a_1 \neq 0 \), then

\[ m_G^{(k)}(a, z) = g_G(a, z) = m_E^2(F(a), F(z)) = \max\{m_E(a_1 a_2, z_1 z_2), m_E(a_1 a_3, z_1 z_3)\}, \quad z \in G, \quad k \in \mathbb{N}. \]

(c) By Example 1.7.13, if \( a_3 \neq 0 \), then

\[ m_G^{(2p)}((0, 0, a_3), z) = g_G((0, 0, a_3), z) = \max\{|z_1 z_2|^{1/2}, |z_1 z_3|\}, \quad z \in G, \quad p \in \mathbb{N}. \]

Moreover,

\[ m_G^{(2p+1)}((0, 0, a_3), z) = \max\{|z_1 z_2|^{\frac{p+1}{2p+1}}, |z_1 z_3|^{\frac{p+1}{2p+1}}, |z_1 z_3|^{\frac{1}{2p+1}}\}, \quad z \in G, \quad p \in \mathbb{Z}_+. \]

Indeed, the inequality “\( \geq \)” is obvious. Thus we have to show that

\[ L(z) := m_G^{(2p+1)}((0, 0, a_3), z) \leq \max\{|z_1 z_2|^{\frac{p+1}{2p+1}}, |z_1 z_2|^{\frac{1}{2p+1}}, |z_1 z_3|^{\frac{1}{2p+1}}, |z_1 z_3|\} =: R(z). \quad (1.7.13) \]

The inequality is obviously true if \( z_1 = 0 \) (because \( \{0\} \times \mathbb{C}^2 \subset G \)). Take \( z = (z_1, z_2, z_3) \in G \) with \( z_1 \neq 0 \). Then

\[ R(z) = \begin{cases} |z_1 z_2|^{\frac{p+1}{2p+1}} \text{ if } |z_2| \geq |z_3| \\ |(z_1 z_2)^p z_1 z_3|^{\frac{1}{2p+1}} \text{ if } |z_1 z_3^p| \leq |z_2| \leq |z_3| \\ |z_1 z_3| \text{ if } |z_2| \leq |z_1 z_3|^p \end{cases}. \]

Using standard argument, we reduce the proof of (1.7.13) to the cases where \( |z_2| = |z_1 z_2|^p \) or \( |z_2| = |z_3| \). If \( |z_2| = |z_1 z_3|^p \), then we have \( L(z) \leq g_G(a, z) = |z_1 z_3| = R(z) \).

If \( |z_2| = |z_3| \), then

\[ L(z) \leq g_G(a, z) = |z_1 z_2|^{1/2}. \quad (1.7.14) \]
Take an arbitrary $f \in \mathcal{O}(G, E)$ with $\text{ord}_a f \geq 2p + 1$. We know that $f(z) = \bar{f}(z_1z_2, z_1z_3)$, where $\bar{f} \in \mathcal{O}(E^2, E)$ (cf. Example 1.7.11). Inequality (1.7.14) shows that

$$|\bar{f}(\lambda, e^{i\theta}\lambda)| \leq |\lambda|^{p+1/2}, \quad \lambda \in E, \theta \in \mathbb{R}.$$  

Hence

$$|\bar{f}(\lambda, e^{i\theta}\lambda)| \leq |\lambda|^{p+1}, \quad \lambda \in E, \theta \in \mathbb{R},$$

and, therefore, if $|z_2| = |z_3|$, then

$$L(z) \leq |z_1z_2|^\frac{p+1}{2p+1} = R(z).$$

(d) Similar formulae hold at points $(0, a_2, 0)$ with $a_2 \neq 0$.

(e) In the case $a_2a_3 \neq 0$, by Example 1.7.13, we already know that $g_G((0, a_2, a_3), z) \geq \max\{|z_1z_2|, |z_1z_3|\}, \quad z \in G.$

One can prove that

$$g_G((0, a_2, a_3), z) \geq m_G^{(k)}(a, z) \geq \max\{|z_1z_2|, |z_1z_3|, \frac{|a_z z_2 - a_3 z_3|}{|a_2| + |a_3|}, z_{\frac{k/2}{k}}\}, \quad z \in G.$$  

It seems that effective formulae for $m_G^{(k)}((0, a_2, a_3), \cdot)$ and $g_G((0, a_2, a_3), \cdot)$ are not known. Let us mention that it must be

$$m_G^{(k)}((0, a_2, a_3), (e^{i\varphi}z_1, e^{i\psi}z_2, e^{i\theta}z_3)) = m_G^{(k)}((0, a_2, a_3), z),$$

$$g_G((0, a_2, a_3), (e^{i\varphi}z_1, e^{i\psi}z_2, e^{i\theta}z_3)) = g_G((0, a_2, a_3), z), \quad z \in G, \varphi, \psi, \theta \in \mathbb{R},$$

and

$$m_G^{(k)}(a, z) = g_G((0, a_2, a_3), z) = \max\{|z_1z_2|, |z_1z_3|\}, \quad z \in G \cap \{a_3z_2 - a_2z_3 = 0\}.$$  

Similarly as in the one-pole case (cf. [J-P 1993], Proposition 4.2.7(h)), the generalized Green function may be characterized in terms of the Monge–Ampère operator $(dd^c \cdot)^n$.

Theorem 1.7.15 ([Le 1989]). Let $G \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Assume that the set $|p|$ is finite. Then the function $u := g_G(p, \cdot)$ is a unique solution of the following problem:

$$\begin{align*}
&u \in C(\overline{G}, [0, 1]), \log u \in \mathcal{PSH}(G), \\
u & = 1 \text{ on } \partial G, \\
&\forall a \in |p| \exists C(a) > 0 \forall z \in G : u(z) \leq C(a)\|z - a\|^p(a), \\
&(dd^c \log u)^n = 0 \text{ in } G \setminus |p|. \quad (49)
\end{align*}$$

The proof is beyond the scope of this article.

(49) Recall that for a locally bounded function $v \in \mathcal{PSH}(D) \cap \mathbb{C}^n$ we have $(dd^c v)^n = 0$ if $v$ is maximal, i.e. for any domain $D_0 \Subset D$ and for any function $v_0 \in \mathcal{PSH}(D_0)$ upper semicontinuous on $\overline{D_0}$, if $v_0 \leq v$ on $\partial D_0$, then $v_0 \leq v$ in $D_0$; cf. [J-P 1993], Appendix, § MA.
Remark 1.7.16. (a) Recall that even in the case of the single pole Green, the Green function $g_G$ is not symmetric. Thus one can for instance ask whether for a bounded hyperconvex domain $G \subset \mathbb{C}^n$ we have $\lim_{a \to b} g_G(a, z) = 1$ for arbitrary $b \in \partial G$ and $z \in G$. The question is also interesting from the point of view of the boundary behavior of the Bergman function.

D. Coman in [Com 1998] proved that if $G$ is a bounded domain with a plurisubharmonic peak function $\rho$ at a point $b \in \partial G$ (i.e. $\rho \in \mathcal{PSH}(G) \cap C(\overline{G})$, $\rho(b) = 0$, and $\rho(z) < 0$, $z \in \overline{G} \setminus \{b\}$) such that $\rho$ is Hölder continuous at $b$, and

$$\max \left\{ \frac{\log |\rho(z)|}{\log |z-b|} : z \in \mathcal{G}, \ r \leq |z-b| \leq 1/2 \right\} = O \left( \log \log \frac{1}{r} \right), \ \ r \to 0,$$

then $\lim_{a \to b} \inf_{z \in K} g_G(a, z) = 1$ for any compact $K \subset \overline{G} \setminus \{b\}$. In particular, the result is true in the case where $G$ is a pseudohyperconvex domain with smooth boundary and $b$ is of finite type.

G. Herbort in [Her 2000] proved that if $G$ is a bounded hyperconvex with a Hölder continuous bounded plurisubharmonic exhaustion function, then $\lim_{a \to b} \inf_{z \in K} g_G(a, z) = 1$ for any $K \subset G$ and $b \in \partial G$. In particular, the result holds if $G$ is a bounded pseudoconvex domain with $C^\infty$ boundary.

(b) Let $G$ be a bounded strictly hyperconvex domain, i.e. there exist a domain $U \subset \mathbb{C}^n$, $G \subset U$, and a function $\rho \in \mathcal{PSH}(U) \cap C(U)$ such that $G = \{z \in U : \rho(z) < 0\}$. S. Nivoche proved [Niv 1994], [Niv 1995], [Niv 2000] that in this case, for every $a \in G$, we have

$$g_G(a, z) = \lim_{k \to +\infty} m_G^{(k)}(a, z) = \sup_{k \in \mathbb{N}} m_G^{(k)}(a, z), \quad z \in G,$$

$$A_G(a; X) = \lim_{k \to +\infty} \gamma_G^{(k)}(a; X) = \sup_{k \in \mathbb{N}} \gamma_G^{(k)}(a; X), \quad X \in \mathbb{C}^n \setminus P,$$

where $P \subset \mathbb{C}^n$ is phiripolar; in fact, $P = \emptyset$ as it was shown by N. Nikolov in [Nik 2000].

Observe that for an elementary Reinhardt domain $D_\alpha$ of irrational type all the $m_{D_\alpha}^{(k)}$’s vanish and $m_{D_\alpha}^{(k)} \not\to g_{D_\alpha}$; cf. Theorem 1.3.1.

(c) In the case $n = 1$ the above result was generalized by N. Nikolov and W. Zwonek in [Nik-Zwo 2003b], Theorem 2. They proved that if $G \subset \mathbb{C}$ is a domain for which the set of one-point connected components of $\mathbb{C} \setminus G$ is polar, then

$$g_G = \sup_{k \in \mathbb{N}} m_G^{(k)}, \quad A_G = \sup_{k \in \mathbb{N}} \gamma_G^{(k)}.$$

Moreover, they gave an example of a hyperconvex domain $G \subset \mathbb{C}$ for which the above equalities do not hold.

(d) Recently E. A. Poletsky [Pol 2002] proved the following important theorem.

Let $G \subset \mathbb{C}^n$ be a bounded strictly hyperconvex domain and let $u$ be a negative plurisubharmonic function on $G$ with zero boundary values, i.e. $\liminf_{z \to \zeta} u(z) = 0$, $\zeta \in \partial G$. Then there exist functions $p_k : G \to \mathbb{R}_+$, $[p_k]$ finite, $k = 1, 2, \ldots$, such that $\log g_G(p_k, \cdot) \to u$ in $L^1(G)$. Moreover, if $u$ is continuous and $\psi \in \mathcal{C}_0((\infty, 0])$, then

$$\int_G \psi(u(z)) (dd^c \log g_G(p_k, \cdot))^n \to \int_G \psi(u(z)) (dd^c u)^n.$$
Example 1.7.17 ([Car-Wie 2003]). Let $p : E^n \rightarrow \mathbb{R}$ be such that

$$|p| = \{a_1, \ldots, a_N\} \subset E \times \{0\}^{n-1}.$$  

Put $a_j = (c_j, 0, \ldots, 0)$, $k_j := p(a_j)$, $j = 1, \ldots, N$, and assume that $k_1 \geq \cdots \geq k_N$. Then

$$g_{E^n}(p, z) = \prod_{j=1}^{N} a_j^{k_j-k_{j+1}}(z), \quad z \in E^n,$$

where $k_{N+1} := 0$ and

$$u_j(z) := \max\{m_{E}(c_1, z_1) \cdots m_{E}(c_j, z_1), |z_2|, \ldots, |z_n|\}$$

$$= \max\{m_{E}(\{c_1, \ldots, c_j\}, z_1), |z_2|, \ldots, |z_n|\}$$

$$= m_{E^n}(\{a_1, \ldots, a_j\}, z), \quad j = 1, \ldots, N.$$  

Moreover, if $k_1, \ldots, k_N \in \mathbb{N}$, then $m_{E^n}(p, \cdot) = g_{E^n}(p, \cdot)$.

The results extends easily to the case where $|p| = \{a_1, \ldots, a_N\} \subset E \times \{c\}^{n-1} \subset E \times E^{n-1}$. Observe that if $k_1 = \cdots = k_N = 1$, then the above formula coincides with that from Example 1.7.2.

Indeed, let $u := \prod_{j=1}^{N} u_j^{k_j-k_{j+1}}$. Notice that $u$ is continuous on $E^n$, log $u$ is plurisubharmonic, and $u = 1$ on $\partial(E^n)$.

Take $1 \leq s \leq N$ and $z = (z_1, \ldots, z_n)$ in a small neighborhood of $a_s$. Then for $j = s, \ldots, N$, we get

$$u_j(z) \leq \max\{\text{const} \, |z_1-c_s|, |z_2|, \ldots, |z_n|\} \leq \text{const} \, ||z-a_s||.$$  

Consequently,

$$u(z) \leq \text{const} \, \prod_{j=s}^{N} u_j^{k_j-k_{j+1}}(z) \leq \text{const} \, \prod_{j=s}^{N} ||z-a_s||^{k_j-k_{j+1}} = \text{const} \, ||z-a_s||^{k_s}.$$  

Thus $g_{E^n}(p, \cdot) \geq u$. To prove the opposite inequality we consider first the case $n = 2$.

By virtue of Theorem 1.7.15, we only need to verify that the function log $u$ is maximal on $E^2 \setminus \{a_1, \ldots, a_N\}$.

Fix a point $b = (b_1, b_2) \in E^2 \setminus \{a_1, \ldots, a_N\}$. Observe that the functions log $u_j(z)$, $j = 1, \ldots, N$, are maximal on $E^2 \setminus \{a_1, \ldots, a_N\}$ (cf. [Kli 1991], Example 3.1.2).

It is clear that there exists at most one $j_0 \in \{1, \ldots, N\}$ such that

$$m(c_1, b_1) \cdots m(c_{j_0}, b_1) = |b_2|.$$  

Consequently, all the functions log $u_j$ with $j \neq j_0$ are pluriharmonic near $b$. Since

$$\log u = \sum_{j=1}^{N} (k_j - k_{j+1}) \log u_j,$$  

we easily conclude that log $u$ is maximal near $b$.

Now, consider the general case $n \geq 3$. Take a point

$$b = (b_1, \ldots, b_n) \in E^n \setminus \{a_1, \ldots, a_N\}.$$
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and let \( \max\{|b_2|, \ldots, |b_n|\} = |b_{s_0}|. \) If \( b_{s_0} = 0 \) (i.e. \( b_2 = \cdots = b_n = 0 \)), then consider the mapping \( E \ni \lambda \xrightarrow{F} (\lambda, 0, \ldots, 0) \in E^n \) and use Remark 1.6.1(e):

\[
g_{E^n}(p, b) = g_{E^n}(p, F(b_1)) = g_E(p \circ F, b_1) = \prod_{j=1}^{N} |m_E(c_j, b_1)|^{k_j - k_{j+1}} = u(b).
\]

If \( b_{s_0} \neq 0 \), then let \( q_s := b_s/b_{s_0} \in \overline{E}, s = 2, \ldots, n. \) Consider the mapping

\[
E^2 \ni (\lambda, \xi) \xrightarrow{F} (\lambda, q_1 \xi, \ldots, q_n \xi) \in E^n.
\]

Using Remark 1.6.1(e) and the case \( n = 2 \), we get

\[
g_{E^n}(p, b) = g_{E^n}(p, F(b_1, b_{s_0})) \leq g_{E^2}(p \circ F_1, b_1, b_{s_0})
= \prod_{j=1}^{N} \max\{|m_E(c_1, b_1) \cdots m(c_j, b_1), |b_{s_0}|\}|^{k_j - k_{j+1}} = u(b).
\]

**Example 1.7.18.** Using Proposition 1.6.6 and Theorem 1.4.1 we get

\[
g_{E^2}((a, b), (z, w)) = c_{G_2}^*(((a + b, ab), (z + w, zw))
= \max \left\{ m_E \left( \frac{2a_w - (a + b)}{2 - (a + b)}, \frac{2a_w - (z + w)}{2 - (z + w)} \right) : a \in \partial E \right\},
(a, b), (z, w) \in E^2, a \neq b. \quad (1.7.15)
\]

Notice that the above case is not covered by Example 1.7.17.

For any points \((a_1, b_1), (a_2, b_2) \in E^2\) with \( m_E(a_1, a_2) = m_E(b_1, b_2) > 0 \) there exists an \( h \in \text{Aut}(E) \) such that \( h(a_1) = b_2, h(a_2) = b_1, \) and consequently, formula (1.7.15) may be easily extended to such pairs of points.

\[\] In the case where \( 0 < m_E(a_1, a_2) \neq m_E(b_1, b_2) > 0, \) an effective formula for \( g_{E^2}((a_1, b_1), (a_2, b_2)), \cdot \) is still unknown \[\]

Recall that by the Lempert theorem, if \( G \subset \mathbb{C}^n \) is convex, then \( c_G^* = k_G \) and, consequently, all holomorphically contractible families coincide on \( G. \) The following example shows that this is not true in the category of generalized holomorphically contractible families.

**Example 1.7.19 (Due to W. Zwonek).** Let

\[D := \{(z, w) \in \mathbb{C}^2 : |z| + |w| < 1\}, \quad A_t := \{(t, \sqrt{t}), (t, -\sqrt{t})\}, \quad 0 < t \ll 1.\]

Then

\[m_D(A_t, (0, 0)) < g_D(A_t, (0, 0)) < d^\max_D(A_t, (0, 0))\]

for small \( t. \)

Indeed, let \( G := \{(z, w) \in \mathbb{C}^2 : |z| + \sqrt{|w|} < 1\} \) and let \( F : D \rightarrow G, F(z, w) := (z, w^2). \) Note that \( F \) is proper and locally biholomorphic in a neighborhood of \( A_t \). Moreover, \( A_t = F^{-1}(t, t). \)

Using Proposition 1.6.6, we conclude that \( g_D(A_t, (0, 0)) = g_G((t, t), (0, 0)). \)
Observe that \( m_D(A_t, (0, 0)) = m_G((t, t), (0, 0)) \). In fact, the inequality “\( \geq \)” follows from (H) (applied to \( F \)). The opposite inequality may be proved as follows. Let \( f \in \mathcal{O}(D, E) \) be such that \( f|_{A_t} = 0 \). Define
\[
\tilde{f}(z, w) := \frac{1}{2}(f(z, \sqrt{w}) + f(z, -\sqrt{w})), \quad (z, w) \in G.
\]
Note that \( \tilde{f} \) is well defined, \( |\tilde{f}| < 1, \tilde{f}(t, t) = 0 \), \( \tilde{f} \) is continuous, and \( \tilde{f} \) is holomorphic on \( D \cap \{w \neq 0\} \). In particular, \( f \) is holomorphic on \( D \). Consequently, \( |f(0, 0)| = |\tilde{f}(0, 0)| \leq m_G((t, t), (0, 0)) \).

Suppose that \( m_D(A_{t_k}, (0, 0)) = g_D(A_{t_k}, (0, 0)) \) for a sequence \( t_k \searrow 0 \). Then
\[
g_G((t_k, t_k), (0, 0)) = g_D(A_{t_k}, (0, 0)) = m_D(A_{t_k}, (0, 0)) = m_G((t_k, t_k), (0, 0)), \quad k = 1, 2, \ldots.
\]
Thus \( m_G((t_k, t_k), (0, 0)) = g_G((t_k, t_k), (0, 0)), k = 1, 2, \ldots \).

Consequently, using [J-P 1993], § 2.5, and [Zwo 2000c], Corollary 4.4 (cf. § 1.2), we conclude that
\[
\gamma_G((0, 0); (1, 1)) = A_G((0, 0); (1, 1)),
\]
where \( \gamma_G \) (resp. \( A_G \)) denotes the Carathéodory–Reiffen (resp. Azukawa) metric of \( G \) (cf. § 1.2). Hence, by Propositions 4.2.7 and 2.2.1(d) from [J-P 1993], using the fact that \( D \) is the convex envelope of \( G \), we get
\[
2 = h_D(1, 1) = \gamma_G((0, 0); (1, 1)) = A_G((0, 0); (1, 1)) = h_G(1, 1) = \frac{2}{3 - \sqrt{5}}; \tag{41}
\]
contradiction.

To see the inequality \( g_D(A_t, (0, 0)) < d_D^{\max}(A_t, (0, 0)) \), we may argue as follows.
We know (cf. [Zwo 2000c], Corollary 4.5 \( ^{\text{(42)}} \)) that
\[
g_D(A_t, (0, 0)) = g_G((t, t), (0, 0)) \approx g_G((0, 0), (t, t)) = h_G(t, t) = \frac{2t}{3 - \sqrt{5}}
\]
for small \( t > 0 \). On the other hand
\[
d_D^{\max}(A_t, (0, 0)) = \min\{\tilde{k}_D((t, -\sqrt{t}), (0, 0)), \tilde{k}_D((t, \sqrt{t}), (0, 0))\}
= \min\{h_D(t, -\sqrt{t}), h_D(t, \sqrt{t})\} = t + \sqrt{t}.
\]

It remains to observe that \( \frac{2t}{3 - \sqrt{5}} < t + \sqrt{t} \) for small \( t > 0 \).

**Example 1.7.20.** Let \( G := E^2, a_- := (-\frac{1}{2}, 0), a_+ := (\frac{1}{2}, 0), b := (0, \frac{1}{2}), p := 2\chi_{a_-} + \chi_{a_+}. \)

Then \( d_{E^2}^{\min}(p, b) \leq d_{E^2}^{\prime}(p, b) < m_{E^2}(p, b) \), where \( d_{E^2}^{\prime}(p, \cdot) \) is defined in Remark 1.7.8. Recall that \( d_{E^2}^{\min}(A, \cdot) \equiv m_{E^2}(A, \cdot) \) \( A \subset E^2 \) — Proposition 1.5.4.

\(^{\text{(41)}} \) Recall that \( h_D \) (resp. \( h_G \)) denotes the Minkowski function for \( D \) (resp. \( G \)).

\(^{\text{(42)}} \) Let \( G \subset C^n \) be a bounded hyperconvex domain. Then
\[
\lim_{z' \neq z''} \frac{g_G(z', z'')}{g_G(z'', z')} = 1, \quad a \in G.
\]
Indeed, by Example 1.7.17, 
\[ m_{E^2}(p, b) = u_1(b)u_2(b) = \max \{ \frac{1}{2}, \frac{1}{3} \} \max \{ \frac{1}{2} \cdot \frac{1}{2}, \frac{1}{3} \} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}. \]

On the other side:
\[
\begin{align*}
  d'_{E^2}(p, b) &= \max \{ m_{E^2}(a_-, b) \} m_{E^2}(a_+, b), \quad m_{E^2}(a_-, b, a_+, b) \}^2 \\
  &= \max \{ \max \{ \frac{1}{2}, \frac{1}{3} \}^2 \max \{ \frac{1}{2}, \frac{1}{3} \}, \quad m_{E^2}([-\frac{1}{3}, \frac{1}{3}] \times \{ 0 \}, b) \}^2 \\
  &= \max \{ \frac{1}{8}, \quad \max \{ \frac{1}{2} \cdot \frac{1}{2}, \frac{1}{3} \}^2 \} = \frac{1}{8}.
\end{align*}
\]

1.8. Properties of $d_{G}^{\min}$ and $d_{G}^{\max}$

Remark 1.8.1. If $D \subset \mathbb{C}^n$ is a Liouville domain (i.e. $\mathcal{O}(D, E) \simeq E$), then
\[
  d_{G}^{\min}(p, z, w) \leq d_{G}^{\min}(p', z), \quad (z, w) \in G \times D,
\]
where $p'(z) := \sup (p(z, w) : w \in D)$, $(z, w) \in G$.

Proposition 1.8.2. (a) The functions $d_{G}^{\min}(p, \cdot)$ and $d_{G}^{\max}(p, \cdot)$ are upper semicontinuous.

(b) If $p : G \rightarrow \mathbb{Z}_+$, then $d_{G}^{\min}(p, \cdot) \in \mathcal{C}(G)$.

Proof. (a) The case of $d_{G}^{\max}(p, \cdot)$ is obvious. To prove the upper semicontinuity of $d_{G}^{\min}(p, \cdot)$, fix a $z_0 \in G$ and suppose that $d_{G}^{\min}(p, z_k) \rightarrow \alpha > \beta > d_{G}^{\min}(p, z_0)$ for a sequence $z_k \rightarrow z_0$. Take functions $f_k \in \mathcal{O}(G, E)$, $k \in \mathbb{N}$, such that $f_k(z_k) = 0$ and $\prod_{\mu \in \mathcal{F}(G)} |\mu|^{\sup p(f_k^{-1}(\mu))} \rightarrow \alpha$. By a Montel argument we may assume that $f_k \rightarrow f_0$ locally uniformly in $G$ with $f_0 \in \mathcal{O}(G, E)$, $f_0(z_0) = 0$. Since $\prod_{\mu \in \mathcal{F}(G)} |\mu|^{\sup p(f_0^{-1}(\mu))} < \beta$, we can find a finite set $A \subset G$ such that $f_0|_A$ is injective and $\prod_{a \in A} |f_0(a)|^{p(a)} < \beta$. Consequently, $\prod_{a \in A} |f_k(a)|^{p(a)} < \beta$ and $f_k|_A$ is injective for $k \gg 1$. Finally, $\prod_{A \in \mathcal{F}(G)} |\mu|^{\sup p(f_k^{-1}(\mu))} < \beta$, $k \gg 1$; contradiction.

(b) In view of (a), it suffices to prove that for every $f \in \mathcal{O}(G, E)$ the function $u_f(z) := \prod_{\mu \in \mathcal{F}(G)} |m_{E}(\mu, f(z))|^{\sup f^{-1}(\mu)}$, $(z, w) \in G$, is continuous on $G$. Observe that $u_f(z) = \inf M \{ \prod_{\mu \in M} |m_{E}(\mu, f(z))|^{\inf k_{\mu}} \}$, where $M$ runs over all finite sets $M \subset f(\{p\})$ such that $k_{\mu} := \sup |\mu|^{-1}(\mu) \subset +\infty \cdot \mu \in M$. Thus $u_f = \inf M \{ |h_{\mu}| \}$, where $h_{\mu} \in \mathcal{O}(G, E)$. Consequently, since the family $(h_{\mu})_M$ is equicontinuous, the function $u_f$ is continuous on $G$.

Example 1.8.3. Let $p : E \times \mathbb{C} \rightarrow \mathbb{R}_+$, $p_{1/k}(k) := \frac{1}{k^2}$, $k = 2, 3, \ldots$, and $p(z, w) := 0$ otherwise. Notice that $|p|$ is discrete. Then by Remark 1.8.1,
\[
  d_{E \times \mathbb{C}}^{\min}(p, (z, w)) = d_{E}^{\min}(p', z) = \prod_{k=2}^{\infty} \max \{ m_{E}(1/k, z) \}^{1/k^2}, \quad (z, w) \in E \times \mathbb{C}.
\]

In particular, $d_{E \times \mathbb{C}}^{\min}(p, \cdot)$ is discontinuous at $(0, w) \in E \times \mathbb{C} \setminus |p|$.
Proposition 1.8.4. If \(||p|| < +\infty\), then for any \(z_0 \in G\) there exists an extremal function for \(d_{G}^{\min}(p, z_0)\), i.e. a function \(f_{z_0} \in \mathcal{O}(G, E)\) with \(f_{z_0}(z_0) = 0\) and

\[
\prod_{\mu \in f_{z_0}(G)} |\mu|^{\sup p(f_{z_0}^{-1}(\mu))} = d_{G}^{\min}(p, z_0).
\]

Proof. Fix a \(z_0 \in G\) and let \(f_k \in \mathcal{O}(G, E)\), \(f_k(z_0) = 0\) be such that

\[
\alpha_k := \prod_{\mu \in f_k(G)} |\mu|^{\sup p(f_k^{-1}(\mu))} \longrightarrow \alpha := d_{G}^{\min}(p, z_0).
\]

Let \(A_k \subset |p|\) be such that \(f_k|A_k\) is injective, \(f_k(A_k) = f_k(|p|)\), and

\[
p(a) = \sup p(f_k^{-1}(f_k(a))), \quad a \in A_k.
\]

Thus \(\alpha_k = \prod_{a \in A_k} |f_k(a)|^{p(a)}\). We may assume that \(A_k = B\) is independent of \(k\) and for any \(a \in B\) the fiber \(B_a := f_k^{-1}(f_k(a)) \cap |p|\) is also independent of \(k\). Moreover, we may assume that \(f_k \to f_0\) locally uniformly in \(G\). Then \(f_0 \in \mathcal{O}(G, E)\), \(f_0(z_0) = 0\), and \(\prod_{a \in B} |f_0(a)|^{p(a)} = \alpha\). Observe that \(f_0(B) = f_0(|p|)\). Let \(B_0 \subset B\) be such that \(f_0|B_0\) is injective and \(f_0(B_0) = f_0(B)\). We have

\[
\alpha \geq \prod_{\mu \in f_0(|p|)} |\mu|^{\sup p(f_0^{-1}(\mu))} = \prod_{\mu \in f_0(B_0)} |\mu|^{\sup p(f_0^{-1}(\mu))} = \prod_{a \in B_0} |f_0(a)|^{\max \{p(b) : b \in B, f_0(b) = f_0(a)\}} \geq \prod_{a \in B} |f_0(a)|^{p(a)} = \alpha.
\]

Proposition 1.8.5. \(\log d_{G}^{\min}(p, \cdot) \in \mathcal{PSH}(G)\).

Proof. By virtue of Proposition 1.8.2(a), we only need to show that for any \(f \in \mathcal{O}(G, E)\) the function \(u_f(z) := \prod_{\mu \in f(G)} |m_{E}(\mu, f(z))|^{\sup p(f^{-1}(\mu))}, \ z \in G\), is log–plurisubharmonic on \(G\). The proof of Proposition 1.8.2 shows that \(u_f = \inf_M v_M\), where \(v_M\) is a log–plurisubharmonic function given by the formula \(v_M(z) := \prod_{\mu \in M} |m_{E}(\mu, f(z))|^{\kappa_f(\mu)}\) and \(M\) runs over a family of finite sets. Observe that \(v_{M_1 \cup M_2} \leq \min\{v_{M_1}, v_{M_2}\}\). It remains to apply Lemma 1.6.3.

Proposition 1.8.6. If \(G_k \not\subset G\) and \(p_k \not\subset p\), then

\[
d_{G_k}^{\min}(p_k, z) \not\subset d_{G}^{\min}(p, z), \quad d_{G_k}^{\max}(p_k, z) \not\subset d_{G}^{\max}(p, z), \quad z \in G.
\]

Proof. By (H) and (M) (Definition 1.5.3) the sequence is monotone and for the limit function \(u\) we have \(u \geq d_{G}^{\min}(p, \cdot)\) (resp. \(u \geq d_{G}^{\max}(p, \cdot)\)). Fix a \(z_0 \in G\).

In the case of the minimal family suppose that \(u(z_0) > \alpha > d_{G}^{\min}(G, z_0)\). Let \(f_k \in \mathcal{O}(G_k, E)\) be such that \(f_k(z_0) = 0\) and \(\prod_{\mu \in f_k(G_k)} |\mu|^{\sup p_k(f_k^{-1}(\mu))} \longrightarrow u(z_0)\). By a Montel argument we may assume that \(f_k \to f_0\) locally uniformly in \(G\) with \(f_0 \in \mathcal{O}(G, E)\), \(f_0(z_0) = 0\). Since \(\prod_{\mu \in f_0(G)} |\mu|^{\sup p(f_0^{-1}(\mu))} < \alpha\), we can find a finite set \(A \subset G\) such that \(f|A\) is injective and \(\prod_{a \in A} |f_0(a)|^{p(a)} < \alpha\). Consequently, \(\prod_{a \in A} |f_k(a)|^{p_k(a)} < \alpha\) and \(f_k|A\) in injective for \(k \gg 1\). Finally, \(\prod_{\mu \in f_k(G_k)} |\mu|^{\sup p_k(f_k^{-1}(\mu))} < \alpha, \ \ k \gg 1\); contradiction.
1. Holomorphically invariant objects

In the case of the maximal family for any \(a \in G\) and \(\varepsilon > 0\) there exists a \(k(a, \varepsilon) \in \mathbb{N}\) such that \(z_0, a \in G_k, \tilde{K}^*_G(a, z_0) \leq K^*_G(a, z_0) + \varepsilon\), and \(p_k(a) \geq p(a) - \varepsilon\) for \(k \geq k(a, \varepsilon)\). Hence

\[
\inf_{k \in \mathbb{N}} d^{\text{max}}_{K^*_G}(p_k, z_0) = \inf_{k \in \mathbb{N}, a \in G_k} [\tilde{K}^*_G(a, z_0)]p_k(a)
\]

\[
\leq \inf_{a \in G} \{[\tilde{K}^*_G(a, z_0) + \varepsilon]p_k(a) : 0 < \varepsilon \ll 1, k \geq k(a, \varepsilon)\}
\]

\[
\leq \inf_{a \in G} \{[\tilde{K}^*_G(a, z_0) + \varepsilon]p(a) - \varepsilon : 0 < \varepsilon \ll 1\} = d^{\text{max}}_{K^*_G}(p, z_0).
\]

\[\square\]

**Example 1.8.7.** Let \(G := \{z \in \mathbb{C}^n : |z^\alpha| < 1\}\), where \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\) is such that \(\alpha_1, \ldots, \alpha_n\) are relatively prime. Then

\[
d^{\text{min}}_{K^*_G}(p, z) = d^{\text{min}}_{E}(p', z^\alpha) = \prod_{\mu \in E} [m_E(\mu, z^\alpha)^{p'(\mu)}], \quad z \in G,
\]

where \(p'(\lambda) = \sup\{p(a) : a^\alpha = \lambda\}, \lambda \in E\).

Indeed, it is known (cf. Example 1.7.9) that any function \(f \in \mathcal{O}(G, E)\) has the form \(f = \tilde{f} \circ F\), where \(F(z) := z^\alpha\) and \(\tilde{f} \in \mathcal{O}(E, E)\). Thus

\[
d^{\text{min}}_{K^*_G}(p, z) = \sup\{ \prod_{\mu \in F(F(G))} [m_E(\mu, \tilde{f}(F(z)))^{\sup p(F^{-1}(\tilde{f}^{-1}(\mu)))} : \tilde{f} \in \mathcal{O}(E, E)\}
\]

\[
\leq \sup\{ \prod_{\mu \in F(E)} [m_E(\mu, \tilde{f}(F(z)))^{\sup p'(\tilde{f}^{-1}(\mu))) : \tilde{f} \in \mathcal{O}(E, E)\} = d^{\text{min}}_{E}(p', F(z)).
\]

1.9. Relative extremal function

**Definition 1.9.1.** Let \(G \subset \mathbb{C}^n\) be a domain. For \(A \subset G\) the relative extremal function of \(A\) in \(G\) is given by the formula (cf. [Klii 1991], § 4.5)

\[
\omega_{A,G} := \sup\{u \in \mathcal{PSH}(G), u \leq 0, u|_A \leq -1\}.
\]

Let \(\omega_{A,G}^*\) denote the upper semicontinuous regularization of \(\omega_{A,G}\).

**Remark 1.9.2.** Let \(F : G \longrightarrow D\) be a holomorphic mapping and let \(A \subset G, B \subset D\) be such that \(F(A) \subset B\). Then

\[
\omega_{B,D}(F(z)) \leq \omega_{A,G}(z), \quad z \in G.
\]

**Theorem 1.9.3** ([Edi 2001]). Let \(G \subset \mathbb{C}^n\) be a domain, let \(p : G \longrightarrow \mathbb{R}_+\) be such that the set \(|p|\) is finite. Fix an \(R > 0\) so small that

- \(\mathbb{B}(a, R^{1/p(\alpha)}) \subset G\) for any \(a \in |p|\),
- \(\mathbb{B}(a, R^{1/p(\alpha)}) \cap \mathbb{B}(b, R^{1/p(\beta)}) = \emptyset\) for any \(a, b \in |p|, a \neq b\).

Let \(A_r := \bigcup_{a \in |p|} \mathbb{B}(a, R^{1/p(\alpha)})\), \(0 < r < R\). Then

\[
\left(\log \frac{R}{r}\right)^{\omega_{A_r,G}} \backslash \log g_G(p, \cdot) \text{ when } r \downarrow 0.
\]
1.9. Relative extremal function

Proof. Let

\[ v_r := \left( \log \frac{R}{r} \right) \omega_{A_r, G}, \quad 0 < r < R. \]

Step 1. \( v_{r_1} \leq v_{r_2} \) for \( 0 < r_1 < r_2 \).

Indeed, fix \( 0 < r_1 < r_2 < R \) and define

\[ u := \frac{v_{r_1}}{\log \frac{R}{r_2}} = \frac{\left( \log \frac{R}{r_1} \right) \omega_{A_{r_1}, G}}{\log \frac{R}{r_2}}. \]

Then \( u \in \mathcal{PSH}(G) \) and \( u \leq 0 \). It suffices to show that \( u \leq -1 \) on \( A_{r_2} \). Fix an \( a \in \|p\| \).

Let \( k := p(a) \). Take a \( z \in B(a, r_1^1/k) \). Then (cf. [Kli 1991], Lemma 4.5.8):

\[ u(z) \leq \frac{\left( \log \frac{R}{r_1} \right) \omega_{B(a, r_1^1/k), B(a, r_1^1/k)}(z)}{\log \frac{R}{r_2}} = \frac{\log R}{\log \frac{R}{r_1}} \left( \log + \frac{\|z-a\|}{R^{1/k}} \frac{\log R}{\log \frac{R}{r_1}} - 1 \right) \leq -1. \]

Let

\[ v := \lim_{r \to 0^+} v_r = \lim_{r \to 0^+} \left( \log \frac{R}{r} \right) \omega_{A_r, G}. \]

Note that \( v \in \mathcal{PSH}(G) \).

Step 2. \( v_r \geq \log g_G(p, \cdot), \quad 0 < r < R \). In particular, \( v \geq \log g_G(p, \cdot) \).

Indeed, fix \( 0 < r < R \) and let

\[ u_r := \frac{\log g_G(p, \cdot)}{\log \frac{R}{r}}. \]

Then \( u_r \in \mathcal{PSH}(G) \) and \( u_r \leq 0 \). Fix a \( a \in \|p\| \) and \( z \in B(a, r_1^1/k) \) (\( k := p(a) \)). Then

\[ u_r(z) \leq \frac{k \log g_{B(a, r_1^1/k)}(a, z)}{\log \frac{R}{r}} = \frac{k \log \frac{\|z-a\|}{R^{1/k}}}{\log \frac{R}{r}} \leq -1. \]

Thus \( u_r \leq \omega_{A_r, G} \).

Step 3. \( v \leq \log g_G(p, \cdot) \).

Indeed, it suffices to check the growth of \( v \) near every point \( a \in \|p\| \). Fix an \( a \in \|p\| \) and let \( z \in B(a, R^{1/k}), \quad z \neq a \) (\( k := p(a) \)). Let \( 0 < r < \|z-a\|^k \). Then

\[ v(z) - k \log \|z-a\| \leq \left( \log \frac{R}{r} \right) \omega_{A_r, G}(z) - k \log \|z-a\| \]

\[ \leq \left( \log \frac{R}{r} \right) \omega_{B(a, r_1^1/k), B(a, r_1^1/k)}(z) - k \log \|z-a\| \]

\[ = \left( \log \frac{R}{r} \right) \left( \frac{\log + \frac{\|z-a\|}{R^{1/k}}}{\log \frac{R}{r_1}} \frac{\log R}{\log \frac{R}{r_1}} - 1 \right) - k \log \|z-a\| \leq -1 \leq -k \log \|z-a\| \leq -\log R. \quad \Box \]
1.10. Analytic discs method

From some general point of view the invariant objects we have studied so far may be divided into three groups:

(a) objects related to certain extremal problems concerning holomorphic mappings $f : G \longrightarrow E$, e.g. $c_G^*(a,z)$, $m_G(a,z)$, $\gamma_G^k(a;X)$, $m_G(p,z)$, $d_G^{\min}(p,z)$;

(b) objects related to certain extremal problems concerning logarithmically plurisubharmonic functions $u : G \longrightarrow [0,1)$, e.g. $g_G(a,z)$, $A_G(a;X)$, $g_G(p,z)$;

(c) objects related to certain extremal problems concerning analytic discs $\varphi : E \longrightarrow G$, e.g. $\tilde{k}_G^*(a,z)$, $H^*(a,z)$, $\kappa_G(a;X)$, $h_G(a;X)$, $\tilde{k}_G^*(p,z)$.

At the end of the eighties E. A. Poletsky invented and partially developed a general method which reduces in some sense problems of type (b) to (c). This method found various important applications, due mainly to A. Edigarian (cf. [Edi 2002] and the references given there) and E. A. Poletsky (cf. [Pol 1991], [Pol 1993], [Edi-Pol 1997]) — see for instance § 1.12. In the present section we are mainly inspired by the exposition of the analytic disc theory presented in [Lár-Sig 1998b] and [Edi 2002].

Definition 1.10.1. Let $G \subset \mathbb{C}^n$ be a domain. By a disc functional (on $G$) we mean any function $\Xi : \mathcal{O}(E,G) \longrightarrow \mathbb{R}$. The envelope of a disc functional $\Xi : \mathcal{O}(E,G) \longrightarrow \mathbb{R}$ is the function $E_\Xi : G \longrightarrow \mathbb{R}$ defined by the formula

$$E_\Xi(z) := \inf\{\Xi(\varphi) : \varphi \in \mathcal{O}(E,G), \varphi(0) = z\}, \quad z \in G.$$ 

Definition 1.10.2. The following four types of disc functionals play important role in complex analysis:

- **Poisson functional:**

  $$\Xi_{\text{Pois}}^p(\varphi) := \frac{1}{2\pi} \int_0^{2\pi} p(\varphi(e^{i\theta})) d\theta, \quad \varphi \in \mathcal{O}(E,G),$$

  where $p : G \longrightarrow [-\infty, \infty)$ is an upper semicontinuous function \(^{(43)}\).

- **Green functional:**

  $$\Xi_{\text{Gre}}^p(\varphi) := \sum_{\lambda \in E_*} p(\varphi(\lambda)) \log |\lambda|, \quad \varphi \in \mathcal{O}(E,G), \quad p : G \longrightarrow \mathbb{R}_+.$$ \(^{(44)}\)

- **Lelong functional:**

  $$\Xi_{\text{Lel}}^p(\varphi) := \sum_{\lambda \in E_*} p(\varphi(\lambda)) \text{ord}_\lambda(\varphi - \varphi(\lambda)) \log |\lambda|, \quad \varphi \in \mathcal{O}(E,G), \quad p : G \longrightarrow \mathbb{R}_+.$$
1.10. Analytic discs method

- **Lempert functional:**
  \[ \mathcal{E}_{\text{Lem}}^p(\varphi) := \inf \{ p(\varphi(\lambda)) \log |\lambda| : \lambda \in E_+ \}, \quad \varphi \in \mathcal{O}(E,G), \quad p : G \rightarrow \mathbb{R}_+. \]

Put \( \mathcal{E}_{\text{Lem}}^p := \mathcal{E}_{\text{Lem}}^p, \mathcal{E}_{\text{Gre}}^p := \mathcal{E}_{\text{Gre}}^p, \mathcal{E}_{\text{Poi}}^p := \mathcal{E}_{\text{Poi}}^p, \mathcal{E}_{\text{Lem}}^p := \mathcal{E}_{\text{Lem}}^p. \)

**Remark 1.10.3.** (a) \( \mathcal{E}_{\text{Lem}}^p \leq \mathcal{E}_{\text{Gre}}^p \leq \mathcal{E}_{\text{Lem}}^p \) and, consequently, \( \mathcal{E}_{\text{Lem}}^p \leq \mathcal{E}_{\text{Gre}}^p \leq \mathcal{E}_{\text{Lem}}^p. \)

(b) Let \( \Xi \in \{ \mathcal{E}_{\text{Gre}}^p, \mathcal{E}_{\text{Lem}}^p \}. \) Then \( \Xi(\varphi) \) is well defined for \( \varphi \in \mathcal{O}(E,G). \) Moreover, \( \mathcal{E}_\Xi(z) = \inf \{ \Xi(\varphi) : \varphi \in \mathcal{O}(E,G), \varphi(0) = z \}, z \in G. \)

Indeed, for \( \varphi \in \mathcal{O}(E,G) \) let \( \varphi_r(\lambda) := \varphi(r\lambda), |\lambda| < 1/r, 0 < r < 1. \) Then \( \varphi_r \in \mathcal{O}(E,G), \varphi_r(0) = \varphi(0), \) and \( \Xi(\varphi) = \inf_{0 < r < 1} \Xi(\varphi_r). \)

(c) If \( F : G \rightarrow D \) is holomorphic, then \( \Xi^p \circ F = \Xi^{q_F} \), \( E \in \{ \Xi_{\text{Poi}}, \Xi_{\text{Gre}}, \Xi_{\text{Lem}} \}, \) and \( \Xi^p \circ F \leq \Xi^{q_F}. \)

(d) Let \( F : G \rightarrow D \) be holomorphic and let \( \Xi : \mathcal{O}(E,D) \rightarrow \mathbb{R} \) be a disc functional on \( D. \) Then the mapping \( \Xi \circ F \) is a disc functional on \( G \) and \( \mathcal{E}_\Xi \circ F \leq \mathcal{E}_{\Xi_{\text{Poi}}} \).

Indeed, we only need to observe that if \( F \) is a covering, then for any disc \( \psi \in \mathcal{O}(E,D) \) with \( \psi(0) = F(z) \) there exists a \( \varphi \in \mathcal{O}(E,G) \) such that \( \varphi(0) = z \) and \( F \circ \varphi = \psi. \)

(e) Observe that \( \hat{k}_G(p,z) := \mathcal{E}_{\text{Lem}}^p(z) = \log \hat{k}_G^*(p,z) = \log d_{\text{out}}^p(z), z \in G \) (cf. § 1.5). In particular, the function \( \hat{k}_G(p, \cdot) \) is upper semicontinuous. Moreover, if \( B(a,r) \subset G, \) then for any \( z \in \partial S(a,r) \) we get

\[ \hat{k}_G(p,z) \leq \log \hat{k}_G^*(a,z) \leq p(a) \log \hat{k}_{B(a,r)}^*(a,z) = p(a) \log \frac{\|z-a\|}{r} = p(a) \log \|z-a\| - p(a) \log r. \]

(f) In the case where \( p = \chi_{(a)} \) we have:

\[ \mathcal{E}_{\text{Poi}}^p(\varphi) = \frac{1}{2\pi} \Lambda(\varphi^{-1}(a) \cap \partial E), \]

\[ \mathcal{E}_{\text{Gre}}^p(\varphi) = \sum_{\lambda \in \varphi^{-1}(a) \cap E_+} \log |\lambda|, \]

\[ \mathcal{E}_{\text{Lem}}^p(\varphi) = \sum_{\lambda \in \varphi^{-1}(a) \cap E_+} \ord_{\lambda} (\varphi - a) \log |\lambda|, \]

\[ \mathcal{E}_{\text{Lem}}^p(\varphi) = \inf \{ \log |\lambda| : \lambda \in \varphi^{-1}(a) \cap E_+ \}, \]

where \( \Lambda \) denotes the Lebesgue measure on \( \partial E \) (\( \Lambda(\partial E) = 2\pi \)).

1.10.1. Poisson functional. For any upper semicontinuous function \( p : G \rightarrow [-\infty, \infty) \) let

\[ \mathcal{P}_p(G) := \{ u \in \mathcal{P}SH(G) : u \leq p \}, \quad \hat{\mathcal{W}}_G(p,z) := \sup \{ u(z) : u \in \mathcal{P}_p(G) \}. \]
The function $\hat{\omega}_G(p, \cdot)$ is called the generalized relative extremal function with weights $p$. Observe that $\hat{\omega}_G(p, \cdot) \in \mathcal{P}_p(G)$. Moreover, $\omega_{U,G} \equiv \hat{\omega}_G(-\chi_U, \cdot)$ for any open set $U \subset G$, where $\omega_{U,G}$ is the relative extremal function (Definition 1.9.1).

**Remark 1.10.4.** (a) If $p_h \searrow p$, then $\mathcal{E}^p_{\text{Poi}} \searrow \mathcal{E}^p_{\text{Poi}}$ and $\mathcal{E}^p_{\text{Poi}} \searrow \mathcal{E}^p_{\text{Poi}}$.

(b) If $p_h \searrow p$, then $\mathcal{E}^p_{\text{Poi}} \searrow \mathcal{E}^p_{\text{Poi}}$ and $\mathcal{E}^p_{\text{Poi}} \searrow \mathcal{E}^p_{\text{Poi}}$.

**Proposition 1.10.5.** For any upper semicontinuous function $p : G \rightarrow [\infty, \infty)$ we have $\hat{\omega}_G(p, \cdot) \leq \mathcal{E}^p_{\text{Poi}}$. Consequently, if $\mathcal{E}^p_{\text{Poi}} \in \mathcal{P}(G)$, then $\mathcal{E}^p_{\text{Poi}} \in \mathcal{P}_p(G)$ and $\hat{\omega}_G(p, \cdot) \equiv \mathcal{E}^p_{\text{Poi}}$.

**Proof.** For $u \in \mathcal{P}_p(G)$ and $\varphi \in \mathcal{O}(E, G)$ we have

$$u(\varphi(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} p(\varphi(e^{i\theta})) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} p(\varphi(e^{i\theta})) d\theta = \mathcal{E}^p_{\text{Poi}}(\varphi).$$

**Lemma 1.10.6.** The function $\mathcal{E}^p_{\text{Poi}}$ is upper semicontinuous on $G$.

**Proof.** By Remark 1.10.4(b), we may assume that $p : G \rightarrow \mathbb{R}$ is continuous. Fix a $z_0 \in G$ and suppose that $\mathcal{E}^p_{\text{Poi}}(z_0) < A$. Then there exists a $\varphi_0 \in \mathcal{O}(E, G)$ such that $\varphi_0(0) = z_0$ and $\mathcal{E}^p_{\text{Poi}}(\varphi_0) < A$. Take $0 < \delta < \text{dist}(\varphi_0(E), \partial G)$. Then for $z \in B(z_0, \delta)$ we get

$$\mathcal{E}^p_{\text{Poi}}(z) \leq \mathcal{E}^p_{\text{Poi}}(\varphi_0 + z - z_0) = \frac{1}{2\pi} \int_0^{2\pi} p(\varphi_0(e^{i\theta}) + z - z_0) d\theta.$$ 

It is clear that the function

$$B(z_0, \delta) \ni z \mapsto \frac{1}{2\pi} \int_0^{2\pi} p(\varphi_0(e^{i\theta}) + z - z_0) d\theta$$

is continuous and smaller than $A$ at $z = z_0$. Consequently, there exists $0 < \delta < r$ such that $\mathcal{E}^p_{\text{Poi}}(z) < A$, $z \in B(z_0, \delta)$.\]

**Theorem 1.10.7.** For any upper semicontinuous function $p : G \rightarrow [\infty, \infty)$ we have $\mathcal{E}^p_{\text{Poi}} \in \mathcal{P}(G)$. Consequently, by Proposition 1.10.5,

$$\inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} p(\varphi(e^{i\theta})) d\theta : \varphi \in \mathcal{O}(E, G), \varphi(0) = z \right\} = \mathcal{E}^p_{\text{Poi}}(z)$$

$$= \hat{\omega}_G(p, z) = \sup \{ u(z) \in \mathcal{P}(G) : u \leq p \}, \quad z \in G.$$ 

In particular, if $U \subset G$ is open, then

$$\omega_{U,G}(z) = \hat{\omega}_G(-\chi_U, z) = \mathcal{E}^p_{-\chi_U}(z)$$

$$= \inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} -\chi_U(\varphi(e^{i\theta})) d\theta : \varphi \in \mathcal{O}(E, G), \varphi(0) = z \right\}$$

$$= -\sup \left\{ \frac{1}{2\pi} A(\{ \xi \in \partial E : \varphi(\xi) \in U \}) : \varphi \in \mathcal{O}(E, G), \varphi(0) = z \right\}, \quad z \in G.$$ 

A class of complex manifolds $G$ for which the above result is true was presented [Lär-Sig 1998b]. The case where $G$ is an arbitrary complex manifold was proved in [Ros 2001], [Edi 2003b].
1.10. Analytic discs method

Proof. By Remark 1.10.4, we may assume that \( p : G \longrightarrow \mathbb{R} \) is continuous. Let \( u_0 := \mathcal{E}^{p}_{\text{Poi}} \).

By Lemma 1.10.6 we only need to show that

\[
u_0(\varphi(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} u_0(\varphi(e^{it})) \, dt, \quad \varphi \in \mathcal{O}(\overline{E}, G).
\]

Fix a \( \varphi_0 \in \mathcal{O}(\overline{E}, G) \). It suffices to prove that for any \( \varepsilon > 0 \) there exists a \( \tilde{\varphi} \in \mathcal{O}(\overline{E}, G) \) such that \( \tilde{\varphi}(0) = \varphi(0) \) and

\[
\Xi_{\text{Poi}}^{\tilde{\varphi}}(\varphi) \leq \frac{1}{2\pi} \int_0^{2\pi} u_0(\varphi_0(e^{it})) \, dt + \varepsilon.
\] (1.10.16)

Fix an \( \varepsilon > 0 \). The proof will be divided into four steps (Lemmas 1.10.8, 1.10.9, 1.10.10, 1.10.11).

Lemma 1.10.8. There exist \( r > 1 \) and \( \Phi \in C^\infty(U_r \times \partial E, G) \), where \( U_r := \mathbb{B}(r) \subset \mathbb{C} \), such that:

i. \( \Phi(\cdot, \xi) \in \mathcal{O}(\overline{E}, G) \), \( \xi \in \partial E \),

ii. \( \Phi(0, \xi) = \varphi_0(\xi) \), \( \xi \in \partial E \),

iii. \( \int_0^{2\pi} \Xi_{\text{Poi}}^{\Phi}(\cdot, e^{it}) \, dt \leq \int_0^{2\pi} u_0(\varphi_0(e^{it})) \, dt + \varepsilon \). (1.10.17)

Proof. Since \( u_0 \) is upper semicontinuous (Lemma 1.10.6), there exists a \( v \in C(G, \mathbb{R}) \) with \( v \geq u_0 \) such that

\[
\int_0^{2\pi} v(\varphi_0(e^{it})) \, dt \leq \int_0^{2\pi} u_0(\varphi_0(e^{it})) \, dt + \frac{\varepsilon}{2}.
\]

For any \( \xi_0 \in \partial E \) there exist \( \varphi \in \mathcal{O}(\overline{E}, G) \), \( 0 < \delta < \operatorname{dist}(\varphi(E), \partial G) \), an open arc \( I \subset \partial E \), and \( r > 1 \) such that:

- \( \xi_0 \in I \), \( \varphi(0) = \varphi_0(\xi_0) \),
- \( \Xi_{\text{Poi}}^{\varphi}(\varphi + z - \varphi_0(\xi_0)) < v(z) + \varepsilon/4 \), \( z \in \mathbb{B}(\varphi_0(\xi_0), \delta) \),
- \( \varphi_0(\xi) \in \mathbb{B}(\varphi_0(\xi_0), \delta), \quad \xi \in I \),
- \( \Phi_0(U_r \times I) \subset G \), where \( \Phi_0(\lambda, \xi) := \varphi(\lambda) + \varphi_0(\xi) - \varphi(0) \).

By a compactness argument we find a finite covering \( \partial E = \bigcup_{r=1}^{N_0} I_r, \ r > 1 \), and functions \( \Phi_\nu \in C^\infty(U_r \times I_\nu, G) \), \( \nu = 1, \ldots, N_0 \), such that:

- \( \Phi_\nu(\cdot, \xi) \in \mathcal{O}(\overline{E}, G) \), \( \xi \in I_\nu \),
- \( \Phi_\nu(0, \xi) = \varphi_0(\xi) \), \( \xi \in I_\nu \),
- \( \Phi_\nu(U_r \times I_\nu) \subset G \),
- \( \Xi_{\text{Poi}}^{\Phi_\nu}(\cdot, \xi) < v(\varphi_0(\xi)) + \varepsilon/4 \), \( \xi \in I_\nu, \nu = 1, \ldots, N_0 \).

Let \( K \) be the closure of the set \( \varphi_0(\partial E) \cup \bigcup_{r=1}^{N_0} \Phi_\nu(U_r \times I_\nu) \) and let \( C > 0 \) be such that \( C > \max\{v(z) : z \in K\} \). There exist disjoint closed arcs \( J_\nu \subset I_\nu, \nu \in A \subset \{1, \ldots, N_0\} \), such that

\[
A(\partial E \setminus \bigcup_{\nu \in A} J_\nu) < \frac{\varepsilon}{C^2}.
\]
We may assume that $A = \{1, \ldots, N\}$ for some $N \leq N_0$. Fix open disjoint arcs $K_\nu$, with $J_\nu \subset K_\nu \subset I_\nu$, $\nu = 1, \ldots, N$, and let $\rho \in C^\infty(\partial E, [0,1])$ be such that $\rho = 1$ on $\bigcup_{\nu=1}^N J_\nu$ and $\text{supp} \rho \subset \bigcup_{\nu=1}^N K_\nu$. Now we define $\Phi: U_\nu \times \partial E \to G$ by the formula
\[
\Phi(\lambda, \xi) := \begin{cases}
\Phi_\nu(\rho(\lambda)\xi), & (\lambda, \xi) \in U_\nu \times K_\nu \\
\varphi_0(\xi), & (\lambda, \xi) \in U_\nu \times (\partial E \setminus \bigcup_{\nu=1}^N K_\nu).
\end{cases}
\]
It is clear that $\Phi$ is well defined, $\Phi \in C^\infty(U_\nu \times \partial E, G)$, $\Phi(U_\nu \times \partial E) \subset K$, and $\Phi$ satisfies (i) and (ii). It remains to check (iii). Let $J_\nu := \{t \in [0,2\pi) : e^{it} \in J_\nu\}$, $\nu = 1, \ldots, N$. We have:
\[
\int_0^{2\pi} \Xi_{\nu}^P(\Phi(\cdot, e^{it}))dt \leq \sum_{\nu=1}^N \int_{J_\nu} \Xi_{\nu}^P(\Phi(\cdot, e^{it}))dt + \frac{\varepsilon}{N}
\]
\[
\leq \sum_{\nu=1}^N \int_{J_\nu} \varepsilon(\phi_0(e^{it}))dt + \frac{\varepsilon}{N} \leq \int_0^{2\pi} \varepsilon(\phi_0(e^{it}))dt + \varepsilon.
\]

**Lemma 1.10.9.** There exists $1 < s < r$ such that for any $j \geq 1$ there exist an open annulus $A_j \supset \partial E$ and $\Phi_j \in \mathcal{O}(U_s \times A_j, G)$ with:

(i) $\Phi_j \to \Phi$ uniformly on $U_s \times \partial E$,
(ii) there exist $1 < s_j < s$ and $k_j \in \mathbb{N}$, $k_j \geq j$, such that the mapping $(\lambda, \xi) \to \Phi_j(\lambda \xi^{k_j}, \xi)$ extends to a mapping $\Psi_j \in \mathcal{O}(U_{s_j} \times U_{s_j}, G)$,
(iii) $\Psi_j(0, \xi) = \varphi_0(\xi)$, $\xi \in U_{s_j}$.

**Proof.** Let
\[
\Phi_j(\lambda, \xi) := \varphi_0(\xi) + \sum_{k=-j}^j \left( \frac{1}{2\pi} \int_0^{2\pi} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta})) e^{-ik\theta} d\theta \right) \xi^k,
\]
observe that the second term is the $j$-th partial sum of the Fourier series of the function $\xi \to \Phi(\lambda, \xi) - \varphi_0(\xi)$; $\Phi_j$ is holomorphic and $\Phi_j(0, \xi) = \varphi_0(\xi)$, $\xi \in (U_s)_\nu$. Moreover, for any $1 < t < r$, $\Phi_j \to \Phi$ uniformly on $U_t \times \partial E$. Indeed, it follows directly from Fourier series theory that $\Phi_j(\lambda, \cdot) \to \Phi(\lambda, \cdot)$ uniformly on $\partial E$ for any $\lambda \in U_r$. Thus we only need to show that the series
\[
\sum_{k=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta})) e^{-ik\theta} d\theta \right) \xi^k
\]
converges uniformly on $U_s \times \partial E$. Using integration by parts, we obtain
\[
\left| \int_0^{2\pi} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta})) e^{-ik\theta} d\theta \right| \xi^k \leq \frac{1}{k^2} \sup_{\lambda \in U, \theta \in [0,2\pi]} \left| \frac{\partial^2}{\partial \theta^2} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta})) \xi^k \right|,
\]
which implies the required convergence.
Fix $1 < t < r$. It follows that $\Phi_j(U_t \times \partial E) \subset G$ for $j \geq j_0$. Hence, one can find an open annulus $A_j \supset \partial E$ such that $\Phi_j(U_t \times A_j) \subset G$, $j \geq j_0$.

For any $\xi \in (U_r)_\times$, the mapping $\Phi_j(\cdot, \xi) = \varphi_0(\xi)$ has a zero at $\lambda = 0$. For any $\lambda \in U_r$ the mapping $\Phi_j(\lambda, \cdot) - \varphi_0$ has a pole of order $\leq j$ at $\xi = 0$. Consequently, for any $k \geq j$ the mapping $(\lambda, \xi) \mapsto \Phi_j(\lambda \xi^k, \xi)$ extends holomorphically to $E \times E$. It remains to check (ii). Recall that $\Phi_j(0, \cdot) = \varphi_0$. Hence there exists $\delta_j > 0$ such that $\Phi_j(U_{\delta_j} \times E) \subset G$.

Since $\Phi_j(U_t \times A_j) \subset G$, $j \geq j_0$, we can find $0 < \rho_j < 1$ such that $\Phi_j(E \times (E \setminus U_{\rho_j})) \subset G$, $j \geq j_0$. Now, let $k_j \geq j$ be so big that $\rho_j^{k_j} < \delta_j$. Then $\Psi_j(\lambda, \xi) := \Phi_j(\lambda \xi^{k_j}, \xi) \in G$, $(\lambda, \xi) \in E \times E$, $j \geq j_0$.

**Lemma 1.10.10.** There exist $1 < s < r$ and $\Psi \in \mathcal{O}(U_s \times U_s, G)$ such that:

(i) $\Psi(0, \xi) = \varphi_0(\xi)$, $\xi \in U_s$,

(ii) \[ \int_0^{2\pi} \Xi_{Poi}(\Psi(\cdot, e^{it}))dt \leq \int_0^{2\pi} \Xi_{Poi}(\Phi(\cdot, e^{it}))dt + \varepsilon. \] (1.10.18)

**Proof.** Let $\Phi_j, \Psi_j$ be as in Lemma 1.10.9. Then, for $j \geq j(\varepsilon)$ we have

\[
\int_0^{2\pi} \Xi_{Poi}(\Psi_j(\cdot, e^{it}))dt = \int_0^{2\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} p(\Phi_j(e^{i(\theta+k_it)}, e^{it}))d\theta \right) dt
\]

\[= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} p(\Phi_j(e^{i\theta}, e^{it}))d\theta dt \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} p(\Phi(e^{i\theta}, e^{it}))d\theta dt + \varepsilon
\]

\[= \int_0^{2\pi} \Xi_{Poi}(\Phi(\cdot, e^{it}))dt + \varepsilon. \] \(\square\)

**Lemma 1.10.11.** There exists a $\theta_0 \in \mathbb{R}$ such that if we put

$\tilde{\varphi}(\lambda) = \varphi_{\theta_0}(\lambda) := \Psi(e^{i\theta_0}\lambda, \lambda)$, $\lambda \in U_s$,

then

$\Xi_{Poi}(\tilde{\varphi}) \leq \frac{1}{2\pi} \int_0^{2\pi} \Xi_{Poi}(\Psi(\cdot, e^{it}))dt.$ (1.10.19)

**Proof.** We have

\[\int_0^{2\pi} \int_0^{2\pi} p(\Psi(e^{i\theta}, e^{it}))d\theta dt = \int_0^{2\pi} \int_0^{2\pi} p(\Psi(e^{i\theta}e^{it}, e^{it}))d\theta d\theta.\]

Consequently, there exists a $\theta_0 \in \mathbb{R}$ such that

\[\Xi_{Poi}(\varphi_{\theta_0}) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} p(\Psi(e^{i\theta_0}e^{it}, e^{it}))d\theta dt \leq \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} p(\Psi(e^{i\theta}e^{it}, e^{it}))d\theta d\theta
\]

\[= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} p(\Psi(e^{i\theta}, e^{it}))d\theta dt = \frac{1}{2\pi} \int_0^{2\pi} \Xi_{Poi}(\Psi(\cdot, e^{it}))dt. \] \(\square\)

Now, using (1.10.19), (1.10.18), and (1.10.17) gives (1.10.16). \(\square\)

The following result is a direct corollary of the definition of the function $\omega_{U,G}$ and Theorem 1.10.7.
Proof. The inequality “≤” follows from Remark 1.9.2. The opposite inequality follows from Theorem 1.10.7 and Remark 1.10.3(c,d):

\[ \omega_{U,G} = \mathcal{E}_{-\chi_{\Omega}} = \mathcal{E}_{-\chi_{\Omega^*}} = \mathcal{E}_{-\chi_{\Omega^*} \circ F} = \mathcal{E}_{-\chi_{\Omega^*}} \circ F = \omega_{V,D} \circ F. \]

\[ \square \]

1.10.2. Green, Lelong, and Lempert functionals. For any function \( p : G \to \mathbb{R}_+ \) let

\[ \mathcal{G}_p(G) := \{ u \in \mathcal{P}\mathcal{SH}(G) : u \leq 0, \forall a \in G \exists C(u,a) \in \mathbb{R} \forall z \in G : u(z) \leq p(a) \log \|z-a\| + C(a) \}. \]

Observe that \( \log g_G(p,z) = \sup \{ u(z) : u \in \mathcal{G}_p(G) \}, z \in G \) (cf. Definition 1.5.1).

Proposition 1.10.13. \( \log g_G(p,\cdot) \leq \mathcal{E}_{\text{Lel}}^p. \) Consequently,

- for \( \mathcal{E} \in \{ \mathcal{E}_\text{Gre}^p, \mathcal{E}_{\text{Lel}}^p, \mathcal{E}_{\text{Lem}}^p \} \) if \( \mathcal{E} \in \mathcal{P}\mathcal{SH}(G) \), then \( \mathcal{E} \in \mathcal{G}_p(G) \) and

\[ \log g_G(p,\cdot) \equiv \mathcal{E}, \]

- if \( \mathcal{E}_{\text{G}(p,\cdot)} \in \mathcal{P}\mathcal{SH}(G) \) \((45)\), then \( \mathcal{E}_{\text{G}(p,\cdot)} \in \mathcal{G}_p(G) \) and

\[ \mathcal{E}_{\text{G}(p,\cdot)} \leq \log g_G(p,\cdot) \leq \mathcal{E}_{\text{Lel}}^p. \]

Proof. Take \( \varphi \in \mathcal{O}(E,G), u \in \mathcal{G}_p(G) \), and a finite set \( B \subset E \cap \varphi^{-1}(\{p\}) \). We are going to prove that \( u(\varphi(0)) \leq \sum_{\lambda \in B} p(\varphi(\lambda)) \text{ord}_\lambda(\varphi - \varphi(\lambda)) \log |\lambda| \), which implies that \( u(\varphi(0)) \leq \mathcal{E}_{\text{Lel}}^p(\varphi) \) and, consequently, \( \log g_G(p,\cdot) \leq \mathcal{E}_{\text{Lel}}^p. \)

Let \( v(\xi) := u(\varphi(\xi)) - \sum_{\lambda \in B} p(\varphi(\lambda)) \text{ord}_\lambda(\varphi - \varphi(\lambda)) \log m_E(\lambda,\xi) \).

Then \( v \in \mathcal{SH}(U_r \setminus B) \) for some \( r > 1 \) and \( v = u \circ \varphi \leq 0 \) on \( \partial E \). Moreover, one can easily check that \( v \) is locally bounded from above in \( U_r \). Hence \( v \) extends subharmonically to \( U_r \) and, by the maximum principle, \( v \leq 0 \) on \( E \). In particular, \( v(0) \leq 0 \), which gives the required inequality. \( \square \)

Proposition 1.10.14. \( \mathcal{E}_\text{Gre}^p = \mathcal{E}_{\text{Lel}}^p. \)

Proof. We have to prove that

\[ L(z) := \inf \left\{ \sum_{\lambda \in E_r} p(\varphi(\lambda)) \text{ord}_\lambda(\varphi - \varphi(\lambda)) \log |\lambda| : \varphi \in \mathcal{O}(E,G), \varphi(0) = z \right\} \]

\[ = \inf \left\{ \sum_{\lambda \in E_r} p(\varphi(\lambda)) \log |\lambda| : \varphi \in \mathcal{O}(E,G), \varphi(0) = z \right\} := R(z), \quad z \in G. \]

The inequality “\( L \leq R \)” is obvious. Fix a \( z \in G \) and an arbitrary constant \( C > L(z) \).

We want to show that \( C \geq R(z) \).

\((45)\) Recall that \( \tilde{k}_G(\varphi,\cdot) = \log \tilde{k}_G^\ast(\varphi,\cdot) \).
It remains to observe that
\[
\sum_{\lambda \in B} p(\varphi(\lambda)) \text{ord}_{\lambda}(\varphi - \varphi(\lambda)) \log |\lambda| < C.
\]
Write \( B = \{b_1, \ldots, b_N\} \), \( a_j := \varphi(b_j) \), \( r(j) := \text{ord}_{b_j}(\varphi - a_j) \), \( j = 1, \ldots, N \). Consider the family of all systems \( c \) of pairwise different points \( c_{j,k} \in E, j = 1, \ldots, N, k = 1, \ldots, r(j) \), such that \( c_{j,1} \cdots c_{j,r(j)} = b_j^{r(j)}, j = 1, \ldots, N \). Define polynomials
\[
Q_{c,\mu,\nu}(\lambda) := \prod_{j=1,\ldots,N}^{k=1,\ldots,r(j)} (\lambda - c_{j,k}), \quad \nu = 1, \ldots, r(\mu),
\]
\[
P_{c,\mu}(\lambda) := \sum_{\nu=1}^{r(\mu)} \frac{Q_{c,\mu,\nu}(\lambda)}{Q_{c,\mu,\nu}(c_{j,\nu})}, \quad \mu = 1, \ldots, N, \lambda \in \mathbb{C}.
\]
Observe that
\begin{itemize}
  \item \( \deg P_{c,j} \leq r(1) + \cdots + r(N) - 1 \),
  \item \( P_{c,j}(c_{\mu,\nu}) = 0 \) if \( \mu \neq j \) and \( P_{c,j}(c_{\mu,\nu}) = 1 \),
  \item \( P_{c,1} + \cdots + P_{c,N} \equiv 1 \).
\end{itemize}
Define
\[
\varphi_c(\lambda) := \sum_{\mu=1}^{N} P_{c,\mu}(\lambda) \left( \frac{\varphi(\lambda) - a_\mu}{(\lambda - b_\mu)^{r(\mu)}} \prod_{j=1,\ldots,N, k=1,\ldots,r(j)} (\lambda - c_{j,k}) \right) + a_\mu.
\]
Observe that \( \varphi_c \in \mathcal{O}(E, \mathbb{C}^n) \), \( \varphi_c(0) = \varphi(0) = z \), and \( \varphi_c(c_{j,k}) = a_j \) for all \( j = 1, \ldots, N, k = 1, \ldots, r(j) \). Moreover,
\[
\sum_{j=1}^{N} \sum_{k=1}^{r(j)} p(a_j) \log |c_{j,k}| = \sum_{j=1}^{N} p(a_j) r(j) \log |b_j| < C.
\]
It remains to observe that \( \varphi_c(E) \subset G \) provided that \( c_{j,k} \approx b_j, j = 1, \ldots, N, k = 1, \ldots, r(j) \). \( \square \)

**Theorem 1.10.15.** If \( |p| \) is finite, then \( E_{E_{\mathbb{D}_G}^p} = E_{\mathbb{D}_G}^p \). Consequently, by Theorem 1.10.7, \( E_{\mathbb{D}_G}^p \in \mathcal{PSH}(G) \) and, therefore, by Propositions 1.10.13 and 1.10.14,
\[
E_{E_{\mathbb{D}_G}^p} = \log g_G(p, \cdot) = E_{\mathbb{D}_G}^p = E_{\mathbb{D}_{\mathbb{C}}^{p_{\mathbb{D}_G}^p}}.
\]
Moreover, by Proposition 1.6.2, for arbitrary \( p : G \to \mathbb{R}_+ \) we get the following Poletsky formula
\[
\log g_G(p, \cdot) = E_{\mathbb{D}_G}^p = E_{\mathbb{D}_{\mathbb{C}}^{p_{\mathbb{D}_G}^p}}.
\]

The Poletsky formula and the main ideas of the proof are due to E. A. Poletsky, cf. [Pol-Sha 1989], [Pol 1991], [Pol 1993]. The first complete proof was given by A. Edigarian in [Edi 1997b]. We follow the exposition of A. Edigarian.
Proof. We may assume that $p \neq 0$. By Proposition 1.10.13 we only need to show that $E_L^p \leq E_{\xi_\nu G}(p)$. Fix a $\varphi_0 \in \mathcal{O}(E, G)$ and $\varepsilon > 0$. It suffices to find a $\tilde{\varphi} \in \mathcal{O}(\overline{E}, G)$ such that $\tilde{\varphi}(0) = \varphi_0(0)$ and

$$E_{\xi_\nu G}(\tilde{\varphi}) \leq \frac{1}{2\pi} \int_0^{2\pi} \tilde{k}_G(p, \varphi_0(e^{it}))dt + \varepsilon.$$ 

The existence of $\varphi$ will be a consequence of a sequence of lemmas (Lemmas 1.10.16 — 1.10.20)

**Lemma 1.10.16.** There exist:
- $1 < s < r$,
- $\Phi \in \mathcal{C}^\infty(U_r \times \partial E, G)$,
- $N \in \mathbb{N}$,
- $a_1, \ldots, a_N \in |p|$,
- $\sigma_1, \ldots, \sigma_N \in \mathcal{C}^\infty(\partial E, C_*)$,
- disjoint closed arcs $J_1, \ldots, J_N \subset \partial E$

such that:
(i) $\Phi(\cdot, \xi) \in \mathcal{O}(\overline{E}, G)$, $\Phi(0, \xi) = \varphi_0(\xi)$, $\xi \in \partial E$,
(ii) if $|\sigma(\xi)| < s$, then $|\sigma(\xi)| > s$, $\mu \neq \nu$, and $\Phi(\sigma(\xi), \xi) = a_\nu$,
(iii) $|\sigma(\xi)| < 1$, $\xi \in J_\nu, \nu = 1, \ldots, N, A(\partial E \cup \bigcup_{\nu=1}^N J_\nu) < \varepsilon$,
(iv) $\sigma(\xi) \neq \sigma(\xi)$, $\xi \in \partial E$, $\nu \neq \mu$,
(v) $2\pi N \max_{\nu=1, \ldots, N} \{p(a_\nu) \max_{\partial E} \log \sigma(\xi)\} < \varepsilon$,
(vi) $\sum_{\nu=1}^N p(a_\nu) \int_0^{2\pi} \log \sigma(\xi)(e^{it})dt \leq \int_0^{2\pi} \tilde{k}_G(p, \varphi_0(e^{it}))dt + \varepsilon$.

Proof. Let $u_0 := \tilde{k}_G(p, \cdot)$. Since $u_0$ is upper semicontinuous, there exists a $v \in \mathcal{C}(G, \mathbb{R})$ with $v \geq u_0$ such that

$$\int_0^{2\pi} v(\varphi_0(e^{it}))dt \leq \int_0^{2\pi} u_0(\varphi_0(e^{it}))dt + \frac{\varepsilon}{2}.$$ 

For any $\xi_0 \in \partial E$ there exist $\varphi \in \mathcal{O}(\overline{E}, G)$, $\lambda_0 \in E_\ast, \delta > 0$, an open arc $I \subset \partial E$, and $r > 1$ such that:
- $\xi_0 \in I, \varphi(0) = \varphi_0(\xi_0), \varphi(\lambda_0) = a \in |p|$,
- $p(a) \log |\lambda_0| < v(z) + \varepsilon/8, \quad z \in B(\varphi(\xi_0), \delta) \subset G$,
- $\varphi(\lambda) + (1 - \lambda/\lambda_0)(z - \varphi(\xi_0)) \in G, \quad (\lambda, z) \in U_r \times B(\varphi(\xi_0), \delta)$,
- $\varphi(\xi) \in B(\varphi(\xi_0), \delta), \quad \xi \in I$.

By a compactness argument we find a covering $\partial E = \bigcup_{\nu=1}^N J_\nu, r > 1$, functions $\Phi_\nu \in \mathcal{C}^\infty(U_r \times I, G), \nu = 1, \ldots, N_0$, and points $\lambda_1, \ldots, \lambda_{N_0} \in E_\ast$ such that:
- $\Phi_\nu(\cdot, \xi) \in \mathcal{O}(\overline{E}, G), \xi \in I_\nu$,
- $\Phi_\nu(0, \xi) = \varphi(\xi), \xi \in I_\nu$,
- $\Phi_\nu(\lambda, \xi) = a_\nu \in |p|, \xi \in I_\nu$,
- $\Phi_\nu(U_r \times I_\nu) \subset G$,
- $p(a_\nu) \log |\lambda_\nu| < v(\varphi_0(\xi)) + \varepsilon/8, \xi \in I_\nu, \nu = 1, \ldots, N_0$.

Replacing $\Phi_\nu$ by the function $(\lambda, \xi) \rightarrow \Phi_\nu(e^{it}\lambda, \xi)$ with suitable $\theta_\nu \approx 0$, we may assume that the points $\lambda_1, \ldots, \lambda_{N_0}$ have different arguments.
1.10. Analytic discs method

Fix $1 < s < s_0 < r$ with $2\pi N_0 \max_{\nu=1,...,N_0} p(a_\nu) \log s_0 < \varepsilon/8$. Let $K$ be the closure of the set

$$\varphi_0(\partial E) \cup \bigcup_{\nu=1}^{N_0} \Phi_\nu(U_r \times I_\nu)$$

and let $C > 0$ be such that

$$C > 2\pi N_0 \max_{\nu=1,...,N_0} p(a_\nu) ||| \lambda_\nu || + \max \{v(z) : z \in K\}.$$

There exist disjoint closed arcs $J_\nu \subset I_\nu$, $\nu \in A \subset \{1, \ldots, N_0\}$, such that

$$A(\partial E \setminus \bigcup_{\nu \in A} J_\nu) < \frac{\varepsilon}{8}.$$ 

We may assume that $A = \{1, \ldots, N\}$ for some $N \leq N_0$. Fix open disjoint arcs $K_\nu$ with $J_\nu \subset K_\nu \subset I_\nu$, $\nu = 1, \ldots, N$, and let $\rho \in C^\infty(\partial E, [0,1])$ be such that $\rho = 1$ on $\bigcup_{\nu=1}^{N_0} J_\nu$ and supp $\rho \subset \bigcup_{\nu=1}^{N} K_\nu$. We define $\Phi : U_r \times \partial E \to G$ by the formula

$$\Phi(\lambda, \xi) := \left\{ \begin{array}{ll} \Phi_\nu(\rho(\xi)\lambda, \xi), & (\lambda, \xi) \in U_r \times K_\nu, \\ \varphi_0(\xi), & (\lambda, \xi) \in U_r \times (\partial E \setminus \bigcup_{\nu=1}^{N} K_\nu). \end{array} \right.$$ 

It is clear that $\Phi$ is well defined, $\Phi \in C^\infty(U_r \times \partial E, G)$, and $\Phi$ satisfies (i).

Let $K_\nu = \{e^{i\theta} : \theta \in (\alpha_\nu, \beta_\nu)\}$, $J_\nu = \{e^{i\theta} : \theta \in [\gamma_\nu, \delta_\nu]\}$ with $\alpha_\nu < \gamma_\nu < \delta_\nu < \beta_\nu$. We may assume that $\rho$ increases on $(\alpha_\nu, \gamma_\nu)$ and decreases on $(\delta_\nu, \beta_\nu)$.

Then the set $J'_\nu := \{\xi \in K_\nu : |\lambda_\nu|/|\rho(\xi)| \leq s\}$ is a closed arc with $J_\nu \subset J'_\nu \subset K_\nu$.

Take a $\sigma_\nu \in C^\infty(\partial E, \mathbb{R} \setminus \{0\})$ with

- $\sigma_\nu(\xi) = \lambda_\nu/|\rho(\xi)|$, $\xi \in J'_\nu$,
- $s < |\sigma_\nu(\xi)| < s_0$, $\xi \in K_\nu \setminus J'_\nu$,
- $|\sigma_\nu(\xi)| = s_0$, $\xi \in \partial E \setminus K_\nu$.

Then (ii), (iii), (iv), and (v) are satisfied. It remains to check (vi). Let $\tilde{J}_\nu := \{\theta \in [0,2\pi) : e^{i\theta} \in J'_\nu\}$, $\nu = 1, \ldots, N$. We have:

$$\sum_{\nu=1}^{N} p(a_\nu) \int_0^{2\pi} \log |\sigma_\nu(e^{i\theta})| d\theta \leq \sum_{\nu=1}^{N} p(a_\nu) \int_{\tilde{J}_\nu} \log |\lambda_\nu| d\theta + \frac{\pi}{8} \leq \sum_{\nu=1}^{N} \int_{J_\nu} v(\varphi_0(e^{i\theta})) d\theta + \frac{\pi}{4} \leq \int_0^{2\pi} v(\varphi_0(e^{i\theta})) d\theta + \frac{\pi}{4} \leq \int_0^{2\pi} \hat{\kappa}_o(p, \varphi_0(e^{i\theta})) d\theta + \varepsilon. \quad \square$$

Lemma 1.10.17. There exists a $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$ there exist $1 < s_j < s$, $\Psi_j \in \mathcal{O}(U_{s_j} \times U_{s_j}, G)$, and $\tau_{\nu,j} \in \mathcal{O}(U_{s_j} \setminus \overline{U_{1/s_j}}, G)$, $\nu = 1, \ldots, N$, such that:

(i) $\Psi_j(0, \xi) = \varphi_0(\xi)$, $\xi \in U_{s_j}$,

(ii) $|\tau_{\nu,j}| \to |\sigma_\nu|$ uniformly on $\partial E$,

(iii) $\Psi_j(\tau_{\nu,j}(\xi), \xi) = a_\nu$, $\xi \in U_{s_j} \setminus \overline{U_{1/s_j}}$ with $|\tau_{\nu,j}(\xi)| < s_j$,

(iv) $|\tau_{\nu,j}(\xi)| < 1$, $\xi \in J_\nu$, $\nu = 1, \ldots, N$. 


Proof. Recall that for any $\xi \in \partial E$ the numbers $0, \sigma_1(\xi), \ldots, \sigma_N(\xi)$ are pairwise different. Let $P : \mathbb{C} \times \partial E \rightarrow \mathbb{C}$ be defined by the formula

$$P(\lambda, \xi) := \varphi_0(\xi) \prod_{\ell=1}^{N} \frac{\lambda - \sigma_\ell(\xi)}{-\sigma_\ell(\xi)} + \sum_{\mu=1}^{N} \frac{\lambda a_\mu}{\sigma_\mu(\xi)} \prod_{\ell=1}^{N} \frac{\lambda - \sigma_\ell(\xi)}{\sigma_\mu(\xi) - \sigma_\ell(\xi)};$$

observe that $P(\cdot, \xi)$ is the Lagrange interpolation polynomial with $P(0, \xi) = \varphi_0(\xi)$, $P(\sigma_\nu(\xi), \xi) = a_\nu$, $\nu = 1, \ldots, N$. We will prove that there exists a function $\Phi_0 \in C^\infty(U_s \times \partial U)$ such that

$$\Phi(\lambda, \xi) = P(\lambda, \xi) + (\lambda - \sigma_1(\xi)) \cdots (\lambda - \sigma_N(\xi))\Phi_0(\lambda, \xi), \quad (\lambda, \xi) \in U_s \times \partial E.$$ 

Indeed, the only problem is to check that $\Phi_0$ is $C^\infty$ near a point $(\sigma_\nu(\xi_0), \xi_0)$ with $|\sigma_\nu(\xi_0)| < s$. Then $|\sigma_\nu(\xi_0)| > s$ for $\mu \neq \nu$, and there exists a neighborhood $V$ of $\xi_0$ such that $|\sigma_\nu(\xi)| < s$, $|\sigma_\mu(\xi)| > s$, $\mu \neq \nu$, for $\xi \in V$. Observe that

$$\Phi(\lambda, \xi) = a_\nu + (\lambda - \sigma_\nu(\xi))\hat{\Phi}(\lambda, \xi),$$

$$P(\lambda, \xi) = a_\nu + (\lambda - \sigma_\nu(\xi))\hat{P}(\lambda, \xi), \quad (\lambda, \xi) \in U_s \times V,$$

where $\hat{\Phi}$ and $\hat{P}$ are $C^\infty$ mappings. Hence

$$\Phi_0(\lambda, \xi) = (\hat{\Phi}(\lambda, \xi) - \hat{P}(\lambda, \xi)) \prod_{\mu \neq \nu} \frac{1}{\lambda - \sigma_\mu(\xi)}, \quad (\lambda, \xi) \in U_s \times V,$$

and, consequently, $\Phi_0 \in C^\infty(U_s \times V)$. Notice that $\Phi_0(0, \cdot) \equiv 0$.

Let $\Phi_{0,j}$ and $\sigma_{\nu,j}$ be the $j$–th partial sums of the Fourier series of $\Phi_0$ and $\sigma_\nu$, respectively, i.e.

$$\Phi_{0,j}(\lambda, \xi) := \sum_{k=-j}^{j} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \Phi_0(\lambda, e^{it}) e^{-ikt} dt \right) \xi^k,$$

$$\sigma_{\nu,j}(\xi) := \sum_{k=-j}^{j} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \sigma_\nu(e^{it}) e^{-ikt} dt \right) \xi^k, \quad (\lambda, \xi) \in U_s \times \mathbb{C};$$

cf. the proof of Lemma 1.10.9. One can easily show that $\Phi_{0,j} \rightarrow \Phi_0$ and $\sigma_{\nu,j} \rightarrow \sigma_\nu$ uniformly on $U_t \times \partial E$ for any $1 < t < s$. Define

$$P_j(\lambda, \xi) := \varphi_0(\xi) \prod_{\ell=1}^{N} \frac{\lambda - \sigma_{\ell,j}(\xi)}{-\sigma_{\ell,j}(\xi)} + \sum_{\mu=1}^{N} \frac{\lambda a_\mu}{\sigma_{\mu,j}(\xi)} \prod_{\ell=1}^{N} \frac{\lambda - \sigma_{\ell,j}(\xi)}{\sigma_{\mu,j}(\xi) - \sigma_{\ell,j}(\xi)};$$

$$\Phi_j(\lambda, \xi) := P_j(\lambda, \xi) + (\lambda - \sigma_{1,j}(\xi)) \cdots (\lambda - \sigma_{N,j}(\xi))\Phi_{0,j}(\lambda, \xi).$$

Then

- $\Phi_{0,j} \in \mathcal{O}(U_s \times \mathbb{C})$,
- for any $\lambda \in U_s$ the function $\Phi_{0,j}(\lambda, \cdot)$ has a pole of order $\leq j$ at $\xi = 0$,
- for any $\xi \in \mathbb{C}_+$ the function $\Phi_{0,j}(\cdot, \xi)$ has a zero at $\lambda = 0$,
- $\Phi_j \rightarrow \Phi$ uniformly on $U_t \times \partial E$, $1 < t < s$,
- $\Phi_j$ is holomorphic on $U_t \times \partial E$, $1 < t < s$, $j \gg 1$. 

• for any \( \lambda \in U_s \) the function \( \Phi_j(\lambda, \cdot) \) has a pole of order \( \leq j \) at \( \xi = 0, j \gg 1 \).

Suppose that \( j \gg 1 \) is such that \( \sigma_{\nu,j}(\xi) \neq 0, \xi \in \partial E \). In particular, the set \( Z_{\nu,j} := E_s \cap \sigma_{\nu,j}^{-1}(0) \) is finite. Observe that \( \Phi_j \in \mathcal{O}(U_s \times (E_s \setminus Z_j)) \), where \( Z_j := \bigcup_{\nu=1}^{N} Z_{\nu,j} \).

Put \( B_j := B_{j_1} \cdots B_{j_N} \), where \( B_{n,j} \) denotes the Blaschke product for \( Z_{\nu,j} \) with the zeros counted with multiplicities \(^{(46)}\). For every \( \xi \in C_s \setminus Z_j \) with \( |\sigma_{\nu,j}(\xi)| < s \), we get \( \Phi_j(\sigma_{\nu,j}(\xi), \xi) = a_\nu \).

For any \( k \geq j \):

• the mapping \( \Phi_{j,k}(\lambda, \xi) := \Phi_j(\lambda \xi^k B_j(\xi), \xi) \) is holomorphic on \( E \times E \) and

• the mapping \( \sigma_{\nu,j,k}(\xi) := \sigma_{\nu,j}(\xi)/(\xi^k B_j(\xi)) \) is meromorphic in \( C_s \) and zero-free holomorphic in \( E_s \).

Moreover, \( \Phi_{j,k}(\sigma_{\nu,j,k}(\xi), \xi) = a_\nu \) for all \( \xi \in (U_s)_* \) such that \( |\sigma_{\nu,j,k}(\xi)\xi^k B_j(\xi)| < s \).

Using the same method as in the proof of Lemma 1.10.9, we get the required result with

\[
\psi_{j}(\lambda, \xi) := \Phi_{j,k}(\lambda, \xi) = \Phi_j(\lambda \xi^k B_j(\xi), \xi),
\]

\[
\psi_{j,k}(\xi) := \sigma_{\nu,j,k}(\xi) = \sigma_{\nu,j}(\xi)/(\xi^k B_j(\xi)),
\]

where \( k_j \geq j \) is sufficiently big (and \( 1 < s_j < s, s_j \approx 1 \)). \( \square \)

Taking in Lemma 1.10.17 a \( j \gg 1 \) gives the following result.

**Lemma 1.10.18.** There exist \( 1 < t < s, \Psi \in \mathcal{O}(U_t \times U_t, G), \tau_\nu \in \mathcal{O}(U_t \setminus \overline{U_{1/t}}, C_s), \\nu = 1, \ldots, N, \) such that:

(i) \( \Psi(0, \xi) = \varphi_0(\xi), \xi \in U_t, \)

(ii) \( |\tau_\nu(\xi)| < 1, \xi \in U_t, \)

(iii) \( \Psi(\tau_\nu(\xi), \xi) = a_\nu, \xi \in (U_s)_* \) with \( |\tau_\nu(\xi)| < t, \nu = 1, \ldots, N, \)

(iv) \( 2\pi N \max_{\nu=1,\ldots,N} \{|p(a_\nu)| \log |\tau_\nu(\xi)| | \leq \varepsilon, \}

(v) \( \sum_{\nu=1}^{N} p(a_\nu) \int_{0}^{2\pi} \log |\tau_\nu(e^{it})| dt \leq \sum_{\nu=1}^{N} p(a_\nu) \int_{0}^{2\pi} \log |\tau_\nu(e^{it})| dt + \varepsilon. \)

**Lemma 1.10.19.** There exist \( \eta_0 \in \partial E, k, c > 0, \) and \( 0 < \rho < 1 \) such that the functions

\[ f(\xi) := \Psi(\eta_0 \xi^k, \xi), \quad F(\lambda, \eta) := \eta \rho^k + e^{-c/k} \]

satisfy

\[ \int_{0}^{2\pi} |\mathcal{P}_t(F(s, e^{it}))| dt \leq \sum_{\nu=1}^{N} p(a_\nu) \int_{0}^{2\pi} \log |\tau_\nu(e^{it})| dt + \varepsilon. \]

**Proof.** Since \( \tau_\nu(\xi) \neq 0, \xi \in U_t \setminus \overline{U_{1/t}}, \) there exists \( \varepsilon > 0 \) such that

\[ \log \left| \frac{\eta e^{-c} - \tau_\nu(\xi)}{1 - \tau_\nu(\xi)\eta e^{-c}} \right| < \log |\tau_\nu(\xi)| + \varepsilon/(2M), \quad \eta, \xi \in \partial E, \nu = 1, \ldots, N, \]

where \( M := \sum_{\nu=1}^{N} p(a_\nu). \) Let

\[ \psi(\lambda) := \exp \left( e \frac{\lambda - 1}{\lambda + 1} \right), \quad \lambda \in C \setminus \{-1\}; \]

observe that \( \psi(E) = E_s, \psi(\partial E \setminus \{-1\}) = \partial E. \)

\(^{(46)}\) That is, the function \( \sigma_{\nu,j}/B_{\nu,j} \) extends to a zero-free holomorphic function on \( E_s. \)
1. Holomorphically invariant objects

Define
\[
\varphi_\nu(\lambda; \eta, \xi) := \frac{\eta \psi(\lambda) - \tau_\nu(\xi)}{1 - \tau_\nu(\xi) \eta \psi(\lambda)}, \quad (\lambda, \eta, \xi) \in (\mathbb{C} \setminus \{-1\}) \times \partial E \times J_\nu;
\]
we have \(|\varphi_\nu(\lambda; \eta, \xi)| = 1, (\lambda, \eta, \xi) \in (\partial E \setminus \{-1\}) \times \partial E \times J_\nu\). Moreover, \(\varphi_\nu(t; \eta, \xi) \to -\tau_\nu(\xi)\) when \(t \to 1^\nu\). Thus \(\varphi_\nu(\cdot; \eta, \xi)\) is an inner function with non-zero radial limits. Consequently, by ..., \(\varphi_\nu(\cdot; \eta, \xi)\) is a Blaschke product. Moreover, since \(\psi'(\lambda) \neq 0\), the zeros of \(\varphi_\nu(\cdot; \eta, \xi)\) are simple and, by the implicit mapping theorem, for any point \((\lambda_0, \eta_0, \xi_0)\) with \(\varphi_\nu(\lambda_0; \eta_0, \xi_0) = 0\), there exists a holomorphic function \(h = h_{\lambda_0, \eta_0, \xi_0}\) defined in neighborhood \(V_0\) of \((\eta_0, \xi_0)\) such that \(h(\eta_0, \xi_0) = \lambda_0\) and \(\varphi_\nu(h(\eta, \xi); \eta, \xi) = 0, (\eta, \xi) \in V_0\). Observe that
\[
h(\eta, \xi) = \frac{1}{2\pi i} \int_{\partial B(\lambda_0, r)} \frac{\lambda \eta \psi'(\lambda)}{\eta \psi(\lambda) - \tau_\nu(\xi)} d\lambda,
\]
where \(B(\lambda_0, r)\) is so small that \(\lambda = \lambda_0\) is the only zero of \(\varphi_\nu(\cdot; \eta_0, \xi_0)\) in \(\partial(\lambda_0, r)\). Let \((\lambda_{\nu, \ell})_{\ell=1}^L\) be the zeros of \(\varphi_\nu(\cdot; \eta_0, \xi_0)\) in \(E\). Since \(\varphi_\nu(\cdot; \eta_0, \xi_0)\) is a Blaschke product, we get
\[
|\varphi_\nu(0; \eta_0, \xi_0)| = \left| \frac{\eta e^{-c} - \tau_\nu(\xi)}{1 - \tau_\nu(\xi) \eta e^{-c}} \right| = \prod_{\ell=1}^L |\lambda_{\nu, \ell}|.
\]
Hence, using (1.10.20), we conclude that there exist \(L \in \mathbb{N}\) and \(\rho > 1\) such that
\[
\sum_{\ell=1}^L \log(|\lambda_{\nu, \ell}|/\rho) < \log |\tau_\nu(\xi_0)| + \varepsilon/(2M).
\]
Consequently,
\[
\sum_{\ell=1}^L \log(|h_{\lambda_{\nu, \ell}, \eta_0, \xi}(\eta, \xi)|/\rho) < \log |\tau_\nu(\xi_0)| + \varepsilon/(2M)
\]
for \((\eta, \xi)\) in a neighborhood of \((\eta_0, \xi_0)\).

Using a compactness argument we see that there exist \(L \in \mathbb{N}\) and \(\rho > 1\) such that for any point \((\eta, \xi) \in \partial E \times J_\nu\), there exist \(\lambda_{\nu, 1}(\eta, \xi), \ldots, \lambda_{\nu, L}(\eta, \xi)\) such that
\[
\varphi_\nu(\lambda_{\nu, \ell}(\eta, \xi)); \eta, \xi) = 0, \quad \ell = 1, \ldots, L,
\]
and
\[
\sum_{\ell=1}^L \log(|\lambda_{\nu, \ell}(\eta, \xi)|/\rho) < \log |\tau_\nu(\xi)| + \varepsilon/(2M).
\]
Let
\[
\psi_k(\lambda) := \frac{\lambda + e^{-c/k}}{1 + e^{-c/k}} = 1 + (1 - e^{-c/k}) \frac{\lambda - 1}{1 + e^{-c/k} \lambda}, \quad \lambda \in \mathbb{C} \setminus \{-e^{-c/k}\}.
\]
Observe that \(\psi_k \to 1\) locally uniformly in \(E\) and
\[
k \log \psi_k(\lambda) \to c \frac{\lambda - 1}{\lambda + 1}, \quad \ell \to \psi_k \to \psi \quad \text{locally uniformly in } E.
\]
Fix a $1 < t_0 < 1/\rho$ and let $V_\rho$ be a neighborhood of $J_\rho$ such that $|\tau_\rho(\xi)| < 1, \xi \in V_\rho$. Let $k_0 \in \mathbb{N}$ be such that $\xi_\psi(\rho\lambda) \in V_\rho, (\lambda, \xi) \in U_{t_0} \times J_\rho, k \geq k_0$. Hence, by (iii) of Lemma 1.10.18, we get

$$
\Psi(\tau_\nu(\xi_\psi(\rho\lambda)), \xi_\psi(\rho\lambda)) = a_\nu, \quad (\lambda, \xi) \in U_{t_0} \times J_\rho, k \geq k_0.
$$

Recall that

$$
\eta\psi^k(\rho\lambda) - \tau_\nu(\xi_\psi(\rho\lambda)) \to \eta\psi(\rho\lambda) - \tau_\nu(\xi)
$$

uniformly with respect to $(\lambda, \eta, \xi) \in U_{t_0} \times \partial E \times J_\rho$. Hence, by the Hurwitz theorem, for $k \gg 1$, there are zeros $\lambda_{\nu, t, k}(\eta, \xi)$ of the function $\lambda \to \eta\psi^k(\rho\lambda) - \tau_\nu(\xi_\psi(\rho\lambda))$ which are so close to $\lambda_{\nu, t}(\eta, \xi)$ that

$$
\sum_{\ell=1}^L \log |\lambda_{\nu, t, k}(\eta, \xi)| < \log |\tau_\nu(\xi)| + \varepsilon/(2M), \quad (\eta, \xi) \in \partial E \times J_\nu.
$$

Observe that

$$
\Psi(\eta\psi^k(\rho\lambda_{\nu, t, k}(\eta, \xi)), \xi_\psi(\rho\lambda_{\nu, t, k}(\eta, \xi))) = a_\nu, \quad (\eta, \xi) \in \partial E \times J_\nu, k \gg 1.
$$

Hence

$$
\Xi_{\text{loc}}^P(\lambda \to \Psi(\eta\psi^k(\rho\lambda), \xi_\psi(\rho\lambda))) < \sum_{\nu=1}^N \mathbf{p}(a_\nu) \log |\tau_\nu(\xi)| + \varepsilon/2,
$$

$$(\eta, \xi) \in Q := \bigcup_{\nu=1}^N(\partial E \times J_\nu).
$$

Consider the diffeomorphism $H : (\partial E)^2 \to (\partial E)^2$ given by $H(\eta, \xi) := (\eta\xi^{-k}, \xi)$. Let $S := H(Q)$. Then $A(S) = A(Q) \geq 2\pi(2\pi - \varepsilon)$ (because the modulus of the Jacobian of $H$ is equal to 1). Consequently, there exists an $\eta_0 \in \partial E$ such that $A(R) \geq 2\pi - \varepsilon$, where $R := \{\xi \in \partial E : (\eta_0, \xi) \in S\}$. We have

$$
\Xi_{\text{loc}}^P(\lambda \to \Psi(\eta_0(\xi_\psi(\rho\lambda))^k, \xi_\psi(\rho\lambda))) \leq \sum_{\nu=1}^N \mathbf{p}(a_\nu) \log |\tau_\nu(\xi)| + \varepsilon/2, \quad \xi \in R.
$$

Finally, by Lemma 1.10.18, we conclude that

$$
\int_0^{2\pi} \Xi_{\text{loc}}^P(\lambda \to \Psi(\eta_0(e^{t\nu}(\rho\lambda))^k, e^{t\nu}\psi(\rho\lambda)))dt \leq \sum_{\nu=1}^N \mathbf{p}(a_\nu) \int_0^{2\pi} \log |\tau_\nu(e^{t\nu})|dt + \varepsilon,
$$

which implies directly the required result. \hfill \square

**Lemma 1.10.20.** There exists a $\theta_0 \in \mathbb{R}$ such that the mapping

$$
\varphi(\xi) := f(F(e^{i\theta_0}\xi, \xi))
$$

satisfies

$$
\Xi_{\text{loc}}^P(\varphi) \leq \frac{1}{2\pi} \int_0^{2\pi} \Xi_{\text{loc}}^P(f(F(\cdot, e^{i\nu})))dt.
$$

(1.10.21)
1. Holomorphically invariant objects

Proof. First we will prove that for any \( \varphi \in \mathcal{O}(\bar{E}, G) \) we have

\[
\Xi_{el}^P(\varphi) = \int_E (\log |\lambda|) \Delta v_\varphi(\lambda) dA_2(\lambda),
\]

where

\[
v_\varphi(\lambda) := \frac{1}{2\pi} \sum_{b \in B_\varphi} \mu(\varphi(b)) \text{ord}_b(\varphi - \varphi(b)) \log m_E(b, \lambda), \quad \lambda \in U_r,
\]

\[
B_\varphi := \{ b \in E_* : \mu(\varphi(b)) > 0 \} = E_* \cap \varphi^{-1}(|\mu|)
\]

(for some \( r > 1 \)). Assume that \( \varphi \) is harmonic near \( \xi \), \( \varphi \in \mathcal{S}(U_r) \). To prove (1.10.22) we use the Riesz representation formula:

\[
\int_E (\log |\lambda|) \Delta v_\varphi(\lambda) dA_2(\lambda) = 2\pi v_\varphi(0) - \int_0^{2\pi} v_\varphi(e^{i\theta}) d\theta = \sum_{b \in B_\varphi} \mu(\varphi(b)) \text{ord}_b(\varphi - \varphi(b)) \log |b| = \Xi_{el}^P(\varphi).
\]

Next we are going to show that for any function \( h \in \mathcal{O}(\bar{E}) \) with \( h(E) \subset E \) we have

\[
\Delta v_{\varphi \circ h} = \Delta (v_\varphi \circ h) \text{ in } E,
\]

with \( \Delta_{-\infty} := 0 \). If \( \varphi \equiv \text{const} \) or \( h \equiv \text{const} \) or \( h(E) \cap B_\varphi = \emptyset \), then (1.10.23) is obviously true. Assume that \( \varphi \not\equiv \text{const} \) and \( h \not\equiv \text{const} \) and \( h(E) \cap B_\varphi \not\supset \emptyset \). It is clear that \( v_{\varphi \circ h} \) and \( v_\varphi \circ h \) are harmonic on \( E \setminus h^{-1}(B_\varphi) \) and, consequently, \( \Delta v_{\varphi \circ h} = \Delta (v_\varphi \circ h) = 0 \) on \( E \setminus h^{-1}(B_\varphi) \).

Take \( b \in B_\varphi \) and \( c \in h^{-1}(b) \). Write \( h(\lambda) = (\lambda - c)^m g(\lambda) \), where \( g \in \mathcal{O}(\bar{E}) \) and \( g(c) \neq 0 \). Then

\[
v_\varphi \circ h(\lambda) = \frac{1}{2\pi} \mu(\varphi(b)) \text{ord}_b(\varphi - \varphi(b)) m \log |\lambda - c| + u(\lambda),
\]

where \( u \) is harmonic near \( c \). Thus

\[
\Delta (v_\varphi \circ h) = \mu(\varphi(b)) \text{ord}_b(\varphi - \varphi(b)) m \delta_c,
\]

where \( \delta_c = \mu(\varphi(h(c))) \text{ord}_c(\varphi - \varphi \circ h(\xi)) \delta_c = \Delta v_{\varphi \circ h} \)

in a neighborhood of \( c \).

Applying (1.10.23) to \( \varphi := f \) and \( h := F(\cdot, \xi) \), we get

\[
\Xi_{el}^P(f(F(\cdot, \xi))) = \int_E (\log |\lambda|) \Delta v_{\varphi \circ F(\cdot, \xi)}(\lambda) dA_2(\lambda)
\]

\[
= \int_E (\log |\lambda|) \Delta (v_\varphi \circ F(\lambda, \xi)) dA_2(\lambda).
\]

Now we need the following auxiliary result.
Lemma 1.10.21. Let \( w \in \mathcal{PSH}(U_r \times U_r) \) \((r > 1)\) and let \( w_\theta(\xi) := w(e^{i\theta}\xi, \xi) \). Then there exists a \( \theta_0 \in \mathbb{R} \) such that
\[
\int_E (\log |\lambda|) \Delta w_\theta(\lambda)dA_2(\lambda) \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \int_E (\log |\lambda|) \Delta w_\lambda(\lambda, e^{i\theta})dA_2(\lambda) \right)d\theta.
\]

Proof. The Riesz representation formula gives:
\[
w(0, 0) = w_\theta(0) = \frac{1}{2\pi} \int_E (\log |\lambda|) \Delta w_\theta(\lambda)dA_2(\lambda) + \frac{1}{2\pi} \int_0^{2\pi} w(e^{i(\theta+i)t}, e^{it})dt.
\]
Hence
\[
2\pi w(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_E (\log |\lambda|) \Delta w_\theta(\lambda)dA_2(\lambda) \right)d\theta
+ \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^{2\pi} w(e^{i(\theta+i)t}, e^{it})dt \right)d\theta.
\]
On the other hand, using the Riesz representation formula for the functions \( w(0, \cdot) \) and \( w(\cdot, e^{i\theta}) \), we get
\[
2\pi w(0, 0) = \int_E (\log |\lambda|) \Delta w_0(\lambda)dA_2(\lambda) + \int_0^{2\pi} w(0, e^{i\theta})d\theta \leq \int_0^{2\pi} w(0, e^{i\theta})d\theta
\]
\[
= \int_0^{2\pi} \left( \frac{1}{2\pi} \int_E (\log |\lambda|) \Delta w_\lambda(\lambda, e^{i\theta})dA_2(\lambda) + \frac{1}{2\pi} \int_0^{2\pi} w(e^{it}, e^{i\theta})dt \right)d\theta.
\]
Consequently,
\[
\int_0^{2\pi} \left( \int_E (\log |\lambda|) \Delta w_\theta(\lambda)dA_2(\lambda) \right)d\theta \leq \int_0^{2\pi} \left( \int_E (\log |\lambda|) \Delta w_0(\lambda)dA_2(\lambda) \right)d\theta,
\]
which implies the required result. \( \square \)

Applying Lemma 1.10.21 to (1.10.24) gives (1.10.21). \( \square \)

Remark 1.10.22. (a) There is a counterpart of the Poletsky formula from Theorem 1.10.15 for the Azukawa pseudometric \( A_G \) (cf. §1.2). Recently, N. Nikolov and W. Zwonek in [Nik-Zwo 2003b], Theorem 1, proved that for any domain \( G \subset \mathbb{C}^n \) we have
\[
A_D = I_G = I_G^1,
\]
where
\[
I_G(a; X) := \inf \left\{ \frac{L_\varphi(a)}{|\ell|} : \varphi \in \mathcal{O}(E, G), \varphi(0) = a, \varphi^{(k)}(0) = k!tX, k := \text{ord}_0(\varphi - a) \right\},
\]
\[
I_G^1(a; X) := \inf \left\{ \frac{L_\varphi(a)}{|\ell|} : \varphi \in \mathcal{O}(E, G), \varphi(0) = a, \varphi'(0) = tX, \text{ord}_0(\varphi - a) = 1 \right\},
\]
\[
L_\varphi(a) := \prod_{\lambda \in \varphi^{-1}(a) \cap E} |\lambda|^{\text{ord}_0(\varphi - a)} = \exp(\Xi_{Le}^{\lambda}(\varphi)), \quad (48) \quad (a, X) \in G \times \mathbb{C}^n.
\]

(48) \( \prod_{\lambda \in \varphi^{-1}(a) \cap E} \cdot := 1 \). See Remark 1.10.3(f) for the definition of \( \Xi_{Le}^{\lambda}(\varphi) \).
1. Holomorphically invariant objects

(b) Let $G \subset \mathbb{C}^n$ be a domain and let $a, z_0 \in G$, $z_0 \neq a$, $X_0 \in \mathbb{C}^*$. Following [Nik-Zwo 2003b], we say that a mapping $\varphi \in \mathcal{O}(E,G)$ is $g_G$-extremal for $(a, z_0)$ (resp. $A_G$-extremal for $(a, X_0)$) if $a, z_0 \in \varphi(E)$, $\varphi(0) = z_0$, and log $g_G(a, z_0) = \Xi_{X_0}^G(\varphi)$ (resp. $\varphi(0) = a$, $\varphi^{(k)}(0) = tk!X_0$, $k := \text{ord}_0(\varphi - a)$, and $A_G(a; X_0) = \frac{k!}{|p|^{(a)}}$) up to an automorphism of $E$ (i.e. we are allowed to substitute $\varphi$ by $\varphi \circ h$, where $h \in \text{Aut}(E)$).

It follows from Proposition 3 in [Nik-Zwo 2003b] that for $\varphi \in \mathcal{O}(E,G)$, $\varphi \neq \text{const}$, $a \in \varphi(E)$, the following conditions are equivalent:

(i) $\varphi$ is $g_G$-extremal for a pair $(a, z_0)$ with $a \neq z_0 \in \varphi(E)$;

(ii) $\varphi$ is $g_G$-extremal for any pair $(a, z)$ with $a \neq z \in \varphi(E)$;

(iii) $\varphi$ is $A_G$-extremal for any pair $(a, \varphi^{(k)}(\lambda))$ with $\lambda \in \varphi^{-1}(a)$, $k := \text{ord}_0(\varphi - a)$.

Moreover, if $G \subset \mathbb{C}$ is such that $\partial G$ is not polar, then a mapping $\varphi \in \mathcal{O}(E,G)$, $a \in \varphi(E)$, $\varphi \neq \text{const}$, is a $g_G$-extremal for some $(a, z_0)$ $(a \neq z_0 \in \varphi(E))$ iff $\varphi = \pi \circ \psi$, where $\pi : E \rightarrow G$ is a universal covering, $\psi \in \mathcal{O}(E,E)$, and the function $h_\lambda \circ \psi = \frac{\psi - \lambda}{1 - \overline{\lambda} \psi}$ is a Blaschke product for any $\lambda \in \pi^{-1}(a)$.

1.11. Coman conjecture

Definition 1.11.1. Let $G$ be a domain in $\mathbb{C}^n$ and let $p : G \rightarrow \mathbb{R}_+$. Define the Coman function

$$\delta_G(p, z) := \inf \left\{ \prod_{a \in |p|} |\mu_a|^{p(a)} : (\mu_a)_{a \in |p|} \subset E, \right.$$  

$$\exists \varphi \in \mathcal{O}(E,G) : \varphi(0) = z, \varphi(\mu_a) = a, a \in |p|, z \in G;$$

we put $\delta_G(p, z) := 1$ if the defining family is empty. We put $\delta_G(A, \cdot) := \delta_G(\chi_A, \cdot)$ $(A \subset G)$, $\delta_G(a, \cdot) := \delta_G(\{a\}, \cdot)$ $(a \in G)$.

Remark 1.11.2. (a) Directly from Proposition 1.10.13 it follows that $g_G(p, \cdot) \leq \delta_G(p, \cdot)$.

(b) Obviously, $\delta_G(a, \cdot) = k_G^a(a, \cdot)$ $(a \in G)$.

(c) $\prod_{a \in |p|} |m_E(a, \cdot)|^{p(a)} = g_E(p, \cdot) = \delta_E(p, \cdot)$ (for any $p : E \rightarrow \mathbb{R}_+$).

Indeed, we only need to prove that $\delta_E(p, \cdot) \leq \prod_{a \in |p|} |m_E(a, \cdot)|^{p(a)}$. Fix a $z_0 \in E$ and let $h := h_{-z_0}$, where $h_{a}(z) := \frac{z - a}{1 - \overline{a}z}$ $(a, z \in E)$. Let $\mu_a := \varphi^{-1}(a)$, $a \in |p|$. Then $\varphi(0) = z_0$, $\varphi(\mu_a) = a$, $a \in |p|$, and $\prod_{a \in |p|} |\mu_a|^{p(a)} = \prod_{a \in |p|} |m_E(\mu_a, 0)|^{p(a)} = \prod_{a \in |p|} |m_E(a, z_0)|^{p(a)}$.

(d) Let $F : G \rightarrow D$ be a holomorphic mapping and let $q : D \rightarrow \mathbb{R}_+$ be such that $\#F^{-1}(b) = 1$ for any $b \in |q|$ (e.g. $F$ is bijective). Then

$$\delta_D(q, F(z)) \leq \delta_G(q \circ F, z), \quad z \in G.$$
Indeed,

$$\delta_D(q, F(z)) = \inf \left\{ \prod_{b \in |q|} |\nu_b|^q(b) : \exists \varphi \in \mathcal{O}(E, D) : \varphi(0) = F(z), \varphi(\mu_b) = b, b \in |q| \right\}$$

$$\leq \inf \left\{ \prod_{a \in F^{-1}(|q|)} |\mu_a|^{q(F(a))} : \exists \varphi \in \mathcal{O}(E, G) : \varphi(0) = z, \varphi(\mu_a) = a, a \in F^{-1}(|q|) \right\}.$$ 

The Coman conjecture says that $g_{E^2}(p, \cdot) \equiv \delta_D(p, \cdot)$ for any convex bounded domain $G$ and function $p$ with $\# |p| < +\infty$ (cf. [Com 2000]). The conjecture was motivated by the Lempert theorem and Remark 1.11.2(b).

D. Coman proved that his conjecture is true in the case where $G = \mathbb{B}_2$ is the unit ball in $C^2$, $|p| = \{a_1, a_2\}$, and $p(a_1) = p(a_2)$ (cf. [Com 2000]).

**Example 1.11.3.** The first counterexample was given by M. Carlehed and J. Wiegerinck in [Car-Wie 2003]: $G = E^2 \subset C^2$, $|p| = \{a_1, a_2\} \subset E \times \{0\}$, $p(a_1) \neq p(a_2)$.

Let $c_1, c_2, d \in E$, $c_1 \neq c_2$, $|c_1| < |c_2|$, $p_{2, 1} := 2\lambda_{c_1,0} + \lambda_{c_2,0}$. Then

$$g_{E^2}(p_{2, 1}, (0, d)) < \delta_{E^2}(p_{2, 1}, (0, d)).$$

Indeed, by Example 1.7.17, 

$$g_{E^2}(p_{2, 1}, c) = \max \{m_{E}(c_1, z_1), |z_2|\} \max \{m_{E}(c_1, z_1)m_{E}(c_2, z_2), |z_2|\}.$$ 

Hence, by Example 1.7.2, if $p_{1, 1} := \lambda_{c_1,0} + \lambda_{c_2,0}$, then

$$\begin{align*}
g_{E^2}(p_{2, 1}, z) &= [g_{E^2}(c_1, 0, z)][g_{E^2}(p_{1, 1}, z)] \\
&\leq [\delta_{E^2}(c_1, 0, z)][\delta_{E^2}(p_{1, 1}, z)] \\
&= [\inf \{|\lambda| : \exists \varphi \in \mathcal{O}(E, E^2) : \varphi(0) = z, \varphi(0) = (c_1, 0)\} \times \\
&\quad \times \inf \{|\lambda_1, \lambda_2| : \exists \varphi \in \mathcal{O}(E, E^2) : \varphi(0) = z, \varphi(0) = (c_1, 0), \varphi(0) = (c_2, 0)\}] \\
&\leq \inf \{|\lambda_1|, |\lambda_2| : \exists \varphi \in \mathcal{O}(E, E^2) : \varphi(0) = z, \varphi(0) = (c_1, 0), \varphi(0) = (c_2, 0)\} \\
&= \delta_{E^2}(p_{2, 1}, z).
\end{align*}$$

Suppose that $g_{E^2}(p_{2, 1}, (0, d)) = \delta_{E^2}(p_{2, 1}, (0, d))$. Then there exist $\varphi_{\nu} \in \mathcal{O}(E, E^2)$ and $\lambda_{\nu, 1}, \lambda_{\nu, 2} \in E$, such that $\varphi_{\nu}(0) = (0, d), \varphi_{\nu}(\lambda_{\nu, 1}) = (c_1, 0), \varphi_{\nu}(\lambda_{\nu, 2}) = (c_2, 0)$, and $|\lambda_{\nu, 1}^2|, |\lambda_{\nu, 2}^2| \rightarrow \max \{m_{E}(c_1, 0), |d|\}|m_{E}(c_1, 0)m_{E}(c_2, 0), |d|\}| = |c_1| d|.

Using Montel argument we easily conclude that there are following three situations:

(a) There exist $\psi_1, \psi_2 \in \mathcal{O}(E, E)$ and $\zeta_1, \zeta_2 \in E$ such that $\psi_1(0) = 0, \psi_2(0) = d$, $\psi_1(\zeta_1) = c_1, \psi_2(\zeta_1) = 0$, $\psi_1(\zeta_2) = c_2, \psi_2(\zeta_2) = 0$, and $|\zeta_1^2| \zeta_2 = |c_1 d|$.

Then, by the Schwarz lemma,

$$|\psi_1(\lambda)| \leq |\lambda|, \quad |\psi_2(\lambda)| \leq \frac{|\lambda - \zeta_1|}{1 - \zeta_1 \lambda}, \quad \frac{|\lambda - \zeta_2|}{1 - \zeta_2 \lambda}, \quad \lambda \in E.$$ 

Hence $|c_1| \leq |\zeta_1|, |d| \leq |\zeta_1 \zeta_2|$ and, consequently, $|c_1| = |\zeta_1|, |d| = |\zeta_1 \zeta_2|$. Thus $|\psi_1(\lambda)| \equiv |\lambda|$. It follows that $|c_2| = |\zeta_2|$ and $|d| = |\zeta_1 \zeta_2| = |c_1 \zeta_2|$, contradiction.

(b) There exist $\psi_1, \psi_2 \in \mathcal{O}(E, E)$ and $\zeta_1 \in E$ such that $\psi_1(0) = 0, \psi_2(0) = d$, $\psi_1(\zeta_1) = c_1, \psi_2(\zeta_1) = 0$, and $|\zeta_1^2| = |c_1 d|$.
Holomorphically invariant objects

Let \( \phi \) be such that \( \phi(0) = 0 \), \( \phi(1) = 1 \), and \( \phi(z) \) is a homeomorphism of \( \mathbb{C} \). Suppose that there exists a sequence \( \{ \phi_k \} \) be such that

\[
\phi_k(0) = (0, \gamma), \quad \phi_k(\xi) = (\sigma_\alpha, \tau), \quad \prod_{\sigma, \tau \in \{-1, +1\}} |\xi_{\sigma, \tau}| \rightarrow a^2.
\]

By Montel argument we may assume that \( \phi_k : E \rightarrow E^2 \) and \( \xi_{\sigma, \tau}^{(k)} \in E \) (\( \sigma, \tau \in \{-1, +1\} \)) be such that

\[
J := \{(\sigma, \tau) \in \{-1, +1\} : \xi_{\sigma, \tau} \in E\}.
\]

Observe that

\[
\prod_{(\sigma, \tau) \in J} |\xi_{\sigma, \tau}| = \prod_{(\sigma, \tau) \in \{-1, +1\}} |\xi_{\sigma, \tau}| = a^2,
\]

in particular, \( J \neq \emptyset \).

It is clear that \( \xi_{\sigma, \tau} \neq \xi_{\sigma', \tau'} \) for \( (\sigma, \tau), (\sigma', \tau') \in J \) with \( \sigma \neq \sigma' \). Put

\[
T := \{\xi_{\sigma, \tau} : (\sigma, \tau) \in J\}.
\]

Let \( \phi_1 := (f_1, g_1) \), \( \phi_2 := (f, g) \). We have \( |g(z)| \leq \prod_{\xi \in T} m_E(\xi, z), \) \( z \in E \). In particular,

\[
\gamma = |g(0)| \leq \prod_{\xi \in T} |\xi|. \quad \text{Consequently, if } \#T = \#J, \text{ then } \gamma \leq a^2 < a^{3/2}; \text{ contradiction.}
\]

From now on assume that \( \#T < \#J \). It suffices to consider the following four cases:

(a) \( \#T = 1, \#J = 2 \): \( J = \{(-1, -1), (-1, 1)\}, \) \( \xi_{-1, -1} = \xi_{-1, 1} = : \xi_{-1} \).

(b) \( \#T = 2, \#J = 3 \): \( J = \{(-1, -1), (-1, 1), (1, -1)\}, \) \( \xi_{-1, -1} = \xi_{-1, 1} = \xi_{1, -1} = \xi_{1, 1} = \xi_{1} \).
1.12. Product property

(c) \#T = 3, \#J = 4: J = \{(-1, -1), (1, 1), (1, -1), (-1, 1)\}, \xi_{-1, -1} = \xi_{-1, 1} =: \xi_{-1}, \\
\xi_{1, -1} = \xi_{1, 1} =: \xi_{1}.

(d) \#T = 2, \#J = 4: J = \{(-1, -1), (1, -1), (1, 1), (-1, 1)\}, \xi_{-1, -1} = \xi_{-1, 1} =: \xi_{-1}, \\
\xi_{1, -1} = \xi_{1, 1} =: \xi_{1}.

Put \( f_{\sigma} := h_{\rho} \circ f = \frac{1 - \rho}{1 - \sigma \rho} f \).

If \((\sigma, \tau) \in J\), then \( f_{\sigma}(\xi_{\sigma, \tau}) = 0\). Hence \( |f_{\sigma}(z)| \leq m_{E}(\xi_{\sigma, \tau}, z), z \in E\). In particular, \( a = |f_{\sigma}(0)| \leq |\xi_{\sigma, \tau}|\).

If \((\sigma, 1), (\sigma, 1) \in J\) and \(\xi_{\sigma, -1} \neq \xi_{\sigma, 1}\), then \( |f_{\sigma}(z)| \leq m_{E}(\xi_{\sigma, -1}, z)m_{E}(\xi_{\sigma, 1}, z), z \in E\). In particular, \( a = |f_{\sigma}(0)| \leq |\xi_{\sigma, -1}| \).

If \((\sigma, 1), (\sigma, 1) \in J\) and \(\xi_{\sigma, -1} = \xi_{\sigma, 1} =: \xi_{\sigma}\), then \( f'(\xi_{\sigma}) = 0\) (if \( f'(\xi_{\sigma}) \neq 0\), then by the Hurwitz theorem, for big \( k\), the equation \( f_{k}(z) = \sigma a\) has exactly one solution in a neighborhood of \( \xi_{\sigma}\), which is false since \( f_{k}(\xi_{\sigma}) = \sigma a, \xi_{\sigma, -1} \neq \xi_{\sigma, 1}\), and \( \xi_{\sigma} \to \xi_{\sigma}\)).

We have \( f_{\sigma}(\xi_{\sigma}) = f'(\xi_{\sigma}) = 0\). Hence \( |f_{\sigma}(z)| \leq |m_{E}(\xi_{\sigma}, z)|^{2}, z \in E\). In particular, \( a = |f_{\sigma}(0)| \leq |\xi_{\sigma}|^{2} \).

Consequently:

In the case (a) we get \( a^{2} = |\xi_{-1}|^{2} \geq a \) — contradiction.

In the case (b) we get \( a^{2} = |\xi_{-1, -1}| \geq a \cdot a\). Hence \( |\xi_{-1}| = |\xi_{-1, -1}| = a\). Since

\( g(z) \leq m_{E}(\xi_{-1}, z)m_{E}(\xi_{-1, -1}, z), z \in E\), we have \( \gamma = |g(0)| \leq |\xi_{-1, -1}| = a^{1/2} \cdot a \) — contradiction.

In the case (c) we get \( a^{2} = |\xi_{-1, -1, -1, 1}| \geq a \cdot a\). Hence \( |\xi_{-1, -1}| = |\xi_{-1, 1}| = a\). Since

\( g(z) \leq m_{E}(\xi_{-1}, z)m_{E}(\xi_{-1, -1}, z)m_{E}(\xi_{1, 1}, z), z \in E\), we have \( \gamma = |g(0)| \leq |\xi_{-1, -1, -1, 1}| = a^{1/2} \cdot a \) — contradiction.

In the case (d) we get \( a^{2} = |\xi_{-1, -1}| \geq a \cdot a\). Hence \( |\xi_{-1}| = |\xi_{-1, 1}| = a\) and, consequently, by the Schwarz lemma, \( h_{\rho} \circ f = f_{\sigma} = \alpha_{\sigma}h_{\xi_{-1}}^{2} \), where \( |\alpha_{\sigma}| = 1, \sigma \in \{-1, +1\}\), which implies that \( f = h_{\sigma}(\alpha_{-1}h_{\xi_{-1}}^{2}) = h_{-a}(\alpha_{1}h_{\xi_{-1}}^{2}) \). In particular, \(-a = f(\xi_{-1}) = h_{-a}(\alpha_{1}h_{\xi_{-1}}^{2}(\xi_{-1}))\) and \( a = f(\xi_{1}) = h_{a}(\alpha_{-1}h_{\xi_{-1}}^{2}(\xi_{1})) \). Then \( \frac{2}{1 + a^{2}} = -\alpha_{1}h_{\xi_{-1}}^{2}(\xi_{1}) = \alpha_{1}h_{\xi_{-1}}^{2}(\xi_{1}) \). Recall that \(-a = f_{\sigma}(0) = \alpha_{\sigma}h_{\xi_{-1}}^{2} \).

Hence \( \frac{2}{1 + a^{2}} = \frac{1}{\xi_{-1}}h_{\xi_{-1}}^{2}(\xi_{-1}) = \frac{1}{\xi_{-1}}h_{\xi_{-1}}^{2}(\xi_{1}). \)

Put \( t := \xi_{-1}/\xi_{1} \). Note that \(|t| = 1\) and \( t \neq 1 \). We have \( \frac{2}{1 + a^{2}} = \frac{(t-1)^{2}}{1 - a^{2}t^{2}} = \frac{(1/1-1)^{2}}{1-1-1/1} \) — contradiction.

**Example 1.11.5.** Let \( D, A_{i} \) be as in Example 1.7.19. Taking \( \varphi(\lambda) := (\lambda^{2}/4, \lambda/2) \), we easily see that \( \delta_{D}(A_{i}, (0, 0)) \leq 4t + v \sqrt{t} = d_{\sigma}^{D}(A_{i}, (0, 0)), 0 < t \ll 1. \)

We do not know whether \( g_{D}(A_{i}, (0, 0)) < \delta_{D}(A_{i}, (0, 0)) \) for small \( t > 0 \) ?

1.12. Product property

1.12.1. Product property for relative extremal function.

**Theorem 1.12.1** ([Edi-Pol 1997], [Edi 2002]). Let \( G_{j} \subset C^{m} \) be a domain, \( A_{j} \subset G_{j}, \)
\( j = 1, 2 \). Assume that \( A_{1}, A_{2} \) are open or \( A_{1}, A_{2} \) are compact. Then

\[ \omega_{A_{1} \times A_{2}, G_{1} \times G_{2}}(z_{1}, z_{2}) = \max\{\omega_{A_{1}, G_{1}}(z_{1}), \omega_{A_{2}, G_{2}}(z_{2})\}, \quad (z_{1}, z_{2}) \in G_{1} \times G_{2}. \]
Moreover, if \(G_1, G_2\) are bounded, then for arbitrary subsets \(A_1 \subset G_1, A_2 \subset G_2\) we have
\[
\omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2) = \max\{\omega_{A_1, G_1}(z_1), \omega_{A_2, G_2}(z_2)\}, \quad (z_1, z_2) \in G_1 \times G_2.
\]

We need a few auxiliary results.

**Proposition 1.12.2** ([Nos 1960], Chapter III). Let \(\varphi \in \mathcal{O}(E, E)\) be an inner function, \(\varphi \not\equiv \text{const}\). Assume that \(\varphi\) is not a Blaschke product. Then there exists a \(\zeta \in \partial E\) such that \(\varphi^*(\zeta) = 0\) \((49)\).

**Proposition 1.12.3** ([Nos 1960], Chapter II). Let \(\varphi \in \mathcal{H}^\infty(E)\) and let \(A \subset \mathbb{C}\) be a compact polar set. Assume that there exists a set \(I \subset \partial E\) of positive measure such that \(\varphi^*(\zeta) \in A, \zeta \in I\). Then \(\varphi \equiv \text{const}\).

**Lemma 1.12.4.** Let \(A \subset E\) be a compact polar set and let \(\pi : E \longrightarrow E \setminus A\) be a universal covering. Then \(\pi\) is an inner function. Moreover, if \(0 \not\in A\), then \(\pi\) is a Blaschke product.

**Proof.** Obviously \(\pi^*(\zeta) \in A \cup \partial E\) for each \(\zeta \in \partial E\) such that \(\pi^*(\zeta)\) exists. Hence, by Proposition 1.12.3, we conclude that \(\pi^*(\zeta) \in \partial E\) for almost all \(\zeta \in \partial E\) and, consequently, \(\pi\) is an inner function. Now, if \(0 \not\in A\), then Proposition 1.12.2 implies that \(\pi\) is a Blaschke product. \(\square\)

**Remark 1.12.5.** Let \(B\) be a finite Blaschke product and let \(\varphi \in \mathcal{O}(E, E)\). Then \(\varphi\) is an inner function iff \(B \circ \varphi\) is inner.

**Lemma 1.12.6** (Löwner theorem, [Edi 2002]). Let \(\varphi \in \mathcal{O}(E, E)\) be an inner function such that \(\varphi(0) = 0\). Then for any open set \(I \subset \partial E\) we have \(A((\varphi^*)^{-1}(I)) = \Lambda(I)\) \((50)\).

**Proof.** We may assume that \(I\) is an arc. Put \(J := (\varphi^*)^{-1}(I)\) (observe that \(J\) is measurable). Consider the following holomorphic functions:
\[
\begin{align*}
  u_I(z) & := \frac{1}{2\pi} \int_0^{2\pi} P(z, \theta)\chi_I(e^{i\theta})d\theta, \\
  u_J(z) & := \frac{1}{2\pi} \int_0^{2\pi} P(z, \theta)\chi_J(e^{i\theta})d\theta, \quad z \in E, \\
  u & := u_I \circ \varphi - u_J,
\end{align*}
\]
where \(P(z, \theta)\) denotes the Poisson kernel. Let \(A\) denote the set of all \(\zeta \in \partial E\) such that:
- \(u_I^*(\zeta)\) does not exist or
- \(u_I^*(\zeta)\) exists but \(u_I^*(\zeta) \not= \chi_I(\zeta)\) or
- \(\varphi^*(\zeta)\) does not exist or
- \(\varphi^*(\zeta)\) exists and \(\varphi^*(\zeta) \in \partial I\) (here \(\partial I\) denotes the boundary of \(I\) in \(\partial E\)).

Note that \(A\) is of zero measure (use Proposition 1.12.3). Observe that \(u^*(\zeta) = 0\) on \(J \setminus A\). Moreover, \(u^*(\zeta) \leq 0\) on \((\partial E \setminus J) \setminus A\). Thus \(u^* \leq 0\) almost everywhere on \(\partial E\) and hence \(u \leq 0\). In particular, \(u(0) = (1/2\pi)(\Lambda(I) - \Lambda(J)) \leq 0\).

Using the same argument to the arc \(\partial E \setminus I\) shows that \(\Lambda(\partial E \setminus I) \leq \Lambda(\partial E \setminus J)\), which finishes the proof. \(\square\)

\((49)\) \(\varphi^*(\zeta) := \lim_{r \to 1} \varphi(r\zeta)\).

\((50)\) Recall that \(A\) denotes the Lebesgue measure on \(\partial E\).
Lemma 1.12.7 ([Edi 2002]). Let \((I_j)_{j=1}^k \subset \partial E\) be a family of disjoint open arcs, let \(I := \bigcup_{j=1}^k I_j\), and let \(\alpha := \Lambda(I)\). Then for every \(\varepsilon > 0\) there exists a finite Blaschke product \(B\) such that:

- \(B(0) = 0\),
- \(B'(z) \neq 0\) for \(z \in B^{-1}(0)\), and
- \(B^{-1}(J_\varepsilon) \subset I\), where \(J_\varepsilon = \{e^{i\theta} : 0 < \theta < \alpha - \varepsilon\}\).

Proof. We may assume that \(\alpha < 2\pi\). Let \(I_j = \{e^{i\theta} : \theta_1 < \theta < \theta_{2,j}\}, j = 1, \ldots, k\), \(J_0 := \{e^{i\theta} : 0 < \theta < \alpha\}\). Define

\[B_0(z) = \frac{\prod_{j=1}^k (z - e^{i\theta_{2,j}}) - e^{i\alpha} \prod_{j=1}^k (z - e^{i\theta_{1,j}})}{\prod_{j=1}^k (z - e^{i\theta_{1,j}}) - \prod_{j=1}^k (z - e^{i\theta_{1,j}})}.
\]

One can prove ([Edi 2002], the proof of Lemma 4.8) that \(B_0\) is a finite Blaschke product with \(B_0(I) = J_0\), \(B_0(\partial E \setminus I) \subset \partial E \setminus J_0\), and \(B_0(\partial E \setminus \overline{I}) = \partial E \setminus \overline{J}_0\).

Suppose that

\[B_0(z) = e^{i\tau} \prod_{j=1}^N \left(\frac{z - a_j}{1 - \overline{a}_j z}\right)^{m_j}.
\]

Take a closed arc \(\overline{J}_0 \subset J_0\) such that \(A(\overline{J}_0) \geq \alpha - \varepsilon\). Then for different points \(a_{j,1}, \ldots, a_{j,m_j}\), sufficiently close to \(a_j\), such that \(a_j \in \{a_{j,1}, \ldots, a_{j,m_j}\}\), if

\[\tilde{B}_0(z) = e^{i\tau} \prod_{j=1}^N \prod_{\ell=1}^{m_j} \left(\frac{z - a_{j,\ell}}{1 - \overline{a}_{j,\ell} z}\right),
\]

then \(\tilde{B}_0(\partial E \setminus I) \subset \partial E \setminus \overline{J}_0\).

Finally, we put \(B(z) := \tilde{B}(e^{i\theta} z)\) (with suitable \(\theta\)). \(\square\)

Proposition 1.12.8 (cf. [Lev-Pol 1999]). Let \(G \subset \mathbb{C}^n\) be a domain and let \(A \subset G\). Then

\[\omega_{A,G} = \sup \{\omega_{U,G} : A \subset U \subset G, U \text{ open}\}.
\]

In particular, if \(A\) is compact, then for any neighborhood basis \((U_j)_{j=1}^\infty\) of \(A\) with \(G \supset U_{j+1} \subset U_j\), we have

\[\omega_{A,G} = \lim_{j \to \infty} \omega_{U_j,G}.
\]

Proof. Let \(u \in \mathcal{PSH}(G), u \leq 0, u \leq -1\) on \(A\). Fix \(0 < \varepsilon < 1\) and define

\[U_\varepsilon := \{z \in G : u < -1 + \varepsilon\}.
\]

Then \(\frac{u}{1-\varepsilon} \leq \omega_{U_\varepsilon,G}\). Consequently,

\[u \leq (1 - \varepsilon) \sup \{\omega_{U,X} : A \subset U \subset X \text{ open}\}.
\]

Taking \(\varepsilon \to 0\), we get the required result. \(\square\)

Proposition 1.12.9 (cf. [Blo 2000]). Let \(G \subset \mathbb{C}^n\) be a bounded domain and let \(A \subset G\). Put \(A_\varepsilon := \{z \in G : \omega_{A,G}(z) < -1 + \varepsilon\}, 0 < \varepsilon < 1\). Then

\[\frac{\omega_{A,G}}{1-\varepsilon} \leq \omega_{A_\varepsilon,G} \leq \omega_{A,G}^*.
\]
Consequently, \( \omega_{A,G} / \omega^*_{A,G} \) as \( \varepsilon \searrow 0 \).

**Proof.** Put \( N := \{ z \in G : \omega_{A,G}(z) < \omega^*_{A,G}(z) \} \) and \( Q := A \setminus N \). It is well-known (see e.g. Theorem 4.7.6 in [Kli 1991]) that \( N \) is pluripolar and \( \omega^*_{Q,G} = \omega^*_{A,G} \) (cf. [Jar-Pfl 2000], Lemma 3.5.3). We have \( \omega^*_{Q,G} = \omega^*_{A,G} = \omega_{A,G} = -1 \) on \( Q \). Hence \( Q \subset A_\varepsilon \) and \( \omega^*_{A,G} = \omega_{A,G} \). Put \( u := \frac{\omega_{A,G}}{1-\varepsilon} \). Note that \( u \in \mathcal{PSH}(G) \), \( u \leq 0 \), and \( u \leq -1 \) on \( A_\varepsilon \). Hence, \( u \leq \omega_{A,G} \).

**Proof of Theorem 1.12.1.** For the proof of the inequality “≥” it suffices to consider projections \( \text{pr}_j : G_1 \times G_2 \to G_j \), \( j = 1, 2 \), and use Remark 1.6.1(e). We move to the opposite inequality.

First assume that \( A_1, A_2 \) are open. Put \( u_1 = -\chi_{A_1} \) and \( u_2 = -\chi_{A_2} \). Let \( (z_1, z_2) \in G_1 \times G_2 \) be fixed and let \( \beta \in \mathbb{R} \) be such that

\[
\max\{\omega_{A_1, G_1}(z_1), \omega_{A_2, G_2}(z_2)\} < \beta.
\]

By Theorem 1.10.7 there are \( \varphi_j \in \mathcal{O}(E, G_j) \), \( j = 1, 2 \), such that \( \varphi_1(0) = z_1 \), \( \varphi_2(0) = z_2 \), and

\[
\frac{1}{2\pi} \int_0^{2\pi} u_j(\varphi_j(e^{i\theta})) d\theta < \beta, \quad j = 1, 2.
\]

Note that \( \varphi_j^{-1}(A_j) \cap \partial E \) is an open set in \( \partial E \). So, we may choose a finite set of disjoint open arcs \( I^1_1, \ldots, I^m_1 \subset \varphi_j^{-1}(A_j) \cap \partial E \) such that \( \Lambda(I^1) > -2\pi \beta \) where \( I^1 = \bigcup_{j=1}^m I^1_j \).

Similarly we choose \( I^1_2, I^2_2 \) with \( I^2 = \bigcup_{j=1}^m I^2_j \). By Lemma 1.12.7 we may find Blaschke products \( B_1, B_2 \) and a closed arc \( I \subset \partial E \) with \( \Lambda(I) > -2\pi \beta \) such that \( B_1^{-1}(I) \subset I^1 \) and \( B_2^{-1}(I) \subset I^2 \).

Let \( A \) be the union of sets of critical values of \( B_1 \) and \( B_2 \). Note that \( 0 \) is not in \( A \). Let \( \pi \) be a holomorphic universal covering of \( E \setminus A \) by \( E \) with \( \pi(0) = 0 \). Observe that \( \pi \) is inner (Lemma 1.12.4). If \( \tilde{I} = \pi^{-1}(I) \), then according to Lemma 1.12.6, \( \Lambda(\tilde{I}) = \Lambda(I) \). There are liftings \( \psi_1 \) and \( \psi_2 \) of \( E \) into \( E \) such that \( \pi = B_1 \circ \psi_1 = B_2 \circ \psi_2 \) and \( \psi_1(0) = \psi_2(0) = 0 \).

By Remark 1.12.5, \( \psi_1 \) and \( \psi_2 \) are inner. Also non-tangential boundary values of \( \psi_1 \) and \( \psi_2 \) on \( \tilde{I} \) belong to \( I^1 \) and \( I^2 \), respectively. Put \( \tilde{\varphi}_1 = \varphi_1 \circ \psi_1 \) and \( \tilde{\varphi}_2 = \varphi_2 \circ \psi_2 \). Then

\[
\frac{1}{2\pi} \int_0^{2\pi} \max\{u_1(\tilde{\varphi}_1(e^{i\theta})), u_2(\tilde{\varphi}_2(e^{i\theta}))\} d\theta \leq -\frac{\Lambda(\tilde{I})}{2\pi} < \beta.
\]

By Fatou’s theorem the same inequality holds if we replace \( \tilde{\varphi}_j(z) \), \( j = 1, 2 \), with \( \tilde{\varphi}_j(rz) \), where \( r < 1 \) is sufficiently close to 1. Hence, \( \omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2) < \beta \). Since \( \beta \) was arbitrary, we get the proof.

The case where \( A_1, A_2 \) are compact follows from Proposition 1.12.8.

We move to the second part of the theorem. First note that for any \( (z_1, z_2) \in G_1 \times G_2 \) we have

\[
\max\{\omega_{A_1, G_1}(z_1), \omega_{A_2, G_2}(z_2)\} \leq \omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2)
\]

\[
\leq -\omega^*_{A_1, G_1}(z_1) \omega^*_{A_2, G_2}(z_2).
\]
Indeed, we only need to prove the second inequality. Let \( u \in \mathcal{PSH}(G_1 \times G_2) \), \( u \leq 0 \), \( u \leq -1 \) on \( A_1 \times A_2 \). Then
\[
\begin{align*}
u(z_1, z_2) & = -\omega_{A_2, G_2}(z_2)\omega_{A_1, G_1}(z_1), \quad z_2 \in A_2, \\
v(z_1, \cdot) & = -\omega_{A_1, G_1}(z_1)\omega_{A_2, G_2}(\cdot), \quad z_1 \in A_1.
\end{align*}
\]
Take a \( z_1 \in G_1 \). If \( \omega_{A_1, G_1}(z_1) = 0 \), then \( u(z_1, \cdot) = 0 = -\omega_{A_1, G_1}(z_1)\omega_{A_2, G_2}(\cdot) \). If \( \omega_{A_1, G_1}(z_1) \neq 0 \), then let \( v := u(z_1, \cdot)/(\omega_{A_1, G_1}(z_1)) \). Then \( v \in \mathcal{PSH}(G_2) \), \( v \leq 0 \), and \( v \leq -1 \) on \( A_2 \). Hence \( v \leq \omega_{A_2, G_2} \).

Fix an \( \varepsilon > 0 \). Then by (1.12.25)
\[
\omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2) \leq -(1 - \varepsilon)^2 \quad \text{on } (A_1)_{\varepsilon} \times (A_2)_{\varepsilon}.
\]
Hence
\[
\omega_{A_1 \times A_2, G_1 \times G_2}^*(z_1, z_2) \leq -(1 - \varepsilon)^2 \quad \text{on } (A_1)_{\varepsilon} \times (A_2)_{\varepsilon}.
\]
It follows that on \( G_1 \times G_2 \)
\[
(1 - \varepsilon)^2 \omega_{A_1 \times A_2, G_1 \times G_2}^* \leq \omega_{(A_1)_{\varepsilon}, G_1 \times G_2} \leq \omega_{A_1 \times A_2, G_1 \times G_2}.
\]
Thus, using the first part of the theorem and Proposition 1.12.9, we get
\[
\omega_{A_1 \times A_2, G_1 \times G_2}^*(z_1, z_2) = \lim_{\varepsilon \to 0} \omega_{(A_1)_{\varepsilon}, (A_2)_{\varepsilon}}\omega_{G_1 \times G_2}(z_1, z_2)
\]
\[
= \lim_{\varepsilon \to 0} \max\{\omega_{(A_1)_{\varepsilon}, G_1}(z_1), \omega_{G_1, (A_2)_{\varepsilon}}(z_2)\}
\]
\[
= \max\{\omega_{A_1, G_1}(z_1), \omega_{A_2, G_2}(z_2)\}, \quad (z_1, z_2) \in G_1 \times G_2. \quad \square
\]


**Theorem.** Let \( G \subset \mathbb{C}^n \) be a convex domain and let \( A \subset G \) be an open or compact convex set. Then for any \( \alpha \in [-1, 0) \) the level set \( \{z \in G : \omega_{A, G}(z) < \alpha\} \) is convex.

1.12.2. Product property for the generalized Green function. Proposition 1.6.2 and Theorems 1.10.15, 1.12.1 imply the following product property for the generalized Green function (cf. [Edi 2001]).

**Theorem 1.12.11.** For any domains \( G_1 \subset \mathbb{C}^n_1, G_2 \subset \mathbb{C}^n_2 \) and for any sets \( A_1 \subset G_1, A_2 \subset G_2 \), the pluricomplex Green function with many poles has the product property:
\[
g_{G_1 \times G_2}(A_1 \times A_2, (z_1, z_2)) = \max\{g_{G_1}(A_1, z_1), g_{G_2}(A_2, z_2)\}, \quad (z_1, z_2) \in G_1 \times G_2.
\]
In particular,
\[
g_{G_1 \times G_2}((a_1, a_2), (z_1, z_2)) = \max\{g_{G_1}(a_1, z_1), g_{G_2}(a_2, z_2)\}, \\
(a_1, a_2), \quad (z_1, z_2) \in G_1 \times G_2.
\]
A different proof, using Polesky’s methods, was given by A. Edigarian in [Edi 1999].

The case of the pluricomplex Green function with one pole has been solved in [Edi 1997a]; some particular cases have been previously solved (using different methods based on the Monge–Ampère operator):
the case where both \( G_1 \) and \( G_2 \) are domains of holomorphy — in [J-P 1993], Theorem 9.8.

the case where at least one of the domains \( G_1, G_2 \) is a domain of holomorphy — in [Jar-Pfl 1995b].

**Remark 1.12.12.** One could try to generalize the above product property to arbitrary pole functions \( p_j : G_j \to \mathbb{R}_+ \) with \( \max_{G_j} p_j = 1, j = 1, 2 \). For instance, one could conjecture that

\[
\delta_{G_1 \times G_2}(p_j(z_1, z_2)) = \max \{\delta_{G_1}(p_1(z_1), \delta_{G_2}(p_2(z_2)) \}, \quad (z_1, z_2) \in G_1 \times G_2,
\]

where \( p(a_1, a_2) := \min \{p_1(a_1), p_2(a_2)\} \).

Unfortunately, such a formula is false.

Take for instance \( G_1 = G_2 = E, p_1 := \chi_{(0)} + \frac{1}{2} \chi_{(c, 0)}, p_2 := \chi_{(0)}, \) where \( 0 < c < 1 \). Observe that \( p = \chi_{((0, 0))} + \frac{1}{2} \chi_{((c, 0))} \). Hence, by Example 1.7.17, we get

\[
g_{E}^{p}(p_j(z_1, z_2)) = \left( \max \{|z_1|, |z_2|\} \max \{|z_1|m(z_1, c)|, |z_2|\}\right)^{1/2}.
\]

In particular, if \( z_1 = c, z_2 = c^2 \), then

\[
g_{E}^{p}(p, (c^2)) = c^{3/2}.
\]

On the other hand,

\[
\max \{g_{E}^{p}(p_1, c), g_{E}^{p}(p_2, c^2)\} = c^2.
\]

**Remark 1.12.13.** Example 1.11.4 shows that, in general, the Coman function does not satisfy the product property. Indeed, let \( B, C, \) and \( \gamma \) be as in the example. Then, by Remark 1.11.2(c), we have

\[
\delta_{E}(B \times C, (0, \gamma)) > g_{E}(B \times C, (0, \gamma))
\]

\[
= \max \{g_{E}(B, 0), g_{E}(C, \gamma)\} = \max \{\delta_{E}(B, 0), \delta_{E}(C, \gamma)\}.
\]

**Proposition 1.12.14 ([Die-Tra 2003]).** For any domains \( G \subset \mathbb{C}^n, D \subset \mathbb{C}^n \), the following conditions are equivalent:

(i) for any finite set \( A \subset G, \) and for any point \( b \in D \) we have

\[
\delta_{G \times D}(A \times \{b\}, (z, w)) = \max \{\delta_{G}(A, z), \delta_{D}(b, w)\}, \quad (z, w) \in G \times D;
\]

(ii) \( \delta_{D}(b, w) = g_{D}(b, w), b, w \in D. \)

**Proof.** (i) \( \Rightarrow \) (ii): By Theorem 1.10.15 we have

\[
g_{D}(b, w) = \inf \{\delta_{D}^{(N)}(b, w), \quad w \in D, \}
\]

where

\[
\delta_{D}^{(N)}(b, w) := \inf \left\{ \prod_{j=1}^{N} |\mu_j| : \mu_1, \ldots, \mu_N \in E, \mu_j \neq \mu_k, \right. \\
\left. \exists \psi \in \mathcal{O}(E, D) : \psi(\mu_j) = b, j = 1, \ldots, N, \psi(0) = w \right\}, \quad w \in D.
\]

By Remark 1.11.2(a), it suffices to show that \( \delta_{D}(b, \cdot) \leq \delta_{D}^{(N)}(b, \cdot) \) (for every \( N \)).
Fix $N \in \mathbb{N}$, $w_0 \in D$, and $\varepsilon > 0$. Let $\mu_1, \ldots, \mu_N \in E$ and $\psi \in \mathcal{O}(E, D)$ be such that $\mu_j \neq \mu_k$, $\psi(\mu_j) = b$, $j = 1, \ldots, N$, $\psi(0) = w_0$, and $\prod_{j=1}^N |\mu_j| \leq \delta(b, w_0) + \varepsilon$.

Take an arbitrary $\varphi \in \mathcal{O}(E, G)$ such that $\varphi(\mu_j) \neq \varphi(\mu_k)$. Put $A := \{\varphi(\mu_1), \ldots, \varphi(\mu_N)\}$, $z_0 := \varphi(0)$. Then

$$\delta_D(b, w_0) \leq \max\{\delta_G(A, z_0), \delta_D(b, w_0)\}$$

$$= \delta_{G \times D}(A \times \{b\}, (z_0, w_0)) \leq \prod_{j=1}^N |\mu_j| \leq \delta_D(b, w_0) + \varepsilon.$$ 

Letting $\varepsilon \to 0$ we conclude the proof.

(ii) $\implies$ (i): Directly from the definition we get the inequality

$$\delta_G(A, z) \leq \delta_{G \times D}(A \times \{b\}, (z, w)), \quad (z, w) \in G \times D.$$ 

Moreover, by Remark 1.11.2(a) and Theorem 1.12.11,

$$\delta_D(b, w) = g_D(b, w) \leq \max\{g_G(A, z), g_D(b, w)\}$$

$$= g_{G \times D}(A \times \{b\}, (z, w)) \leq \delta_{G \times D}(A \times \{b\}, (z, w)), \quad (z, w) \in G \times D.$$ 

Thus

$$\delta_{G \times D}(A \times \{b\}, (z, w)) \geq \max\{\delta_G(A, z), \delta_D(b, w)\}, \quad (z, w) \in G \times D.$$ 

Let $A = \{a_1, \ldots, a_N\}$. Fix $(z_0, w_0) \in G \times D$ and $\varepsilon > 0$. To prove the inequality

$$\delta_{G \times D}(A \times \{b\}, (z_0, w_0)) \leq \max\{\delta_G(A, z_0), \delta_D(b, w_0)\},$$ 

we may assume that $\max\{\delta_G(A, z_0), \delta_D(b, w_0)\} + \varepsilon < 1$. Consider the following two cases:

(a) $\delta_D(b, w_0) \leq \delta_G(A, z_0)$.

Take $\mu_1, \ldots, \mu_N \in E$ and $\varphi \in \mathcal{O}(E, G)$ such that $\varphi(0) = z_0$, $\varphi(\mu_j) = a_j$, $j = 1, \ldots, N$, and $\prod_{j=1}^N |\mu_j| < \delta_G(A, z_0) + \varepsilon$.

We may assume that $\mu_j \neq 0$, $j = 1, \ldots, N$.

Indeed, suppose that $\mu_1 \cdots \mu_{N-1} = 0$, and $\mu_N = 0$. Then we may substitute $\varphi$ by $\tilde{\varphi} := \varphi \circ B$, where $B(z) := \frac{z}{\mu_N}$, $z \in E$. Observe that $B(E) = E$, so there exist $\tilde{\mu}_1, \ldots, \tilde{\mu}_{N-1} \in E$, with $B(\tilde{\mu}_j) = \mu_j$, $j = 1, \ldots, N-1$. Hence $\tilde{\varphi}(\mu_j) = a_j$, $j = 1, \ldots, N - 1$, $\tilde{\varphi}(\varepsilon) = \tilde{\varphi}(0) = z_0$, and $|\mu_1 \cdots \mu_{N-1}\varepsilon| < \varepsilon$.

We may also assume that $\delta_D(b, w_0) < \prod_{j=1}^N |\mu_j|$.

Indeed, if $\delta_D(b, w_0) = \prod_{j=1}^N |\mu_j|$, then we may substitute $\varphi$ by $\tilde{\varphi}(z) := \varphi(tz)$, $z \in E$, and $\mu_j$ by $\mu_j/t$, $j = 1, \ldots, N$, with suitable $0 < t < 1$, $t \approx 1$.

Take $\eta \in E$ and $\psi \in \mathcal{O}(E, D)$ such that $\psi(0) = w_0$, $\psi(\eta) = b$, and $|\eta| < \prod_{j=1}^N |\mu_j|$.

Define $\alpha := \prod_{j=1}^N (-\mu_j) \in E$, $t := -\eta/\alpha \in E$, $\tilde{\psi}(z) := \psi(tz)$, $z \in E$, $B := \prod_{j=1}^N h_{\mu_j}$,

$$\chi : E \to G \times D, \chi := (\varphi, \tilde{\psi} \circ h_{\alpha} \circ B).$$

We have $\chi(0) = (\varphi(0), \tilde{\psi}(h_{\alpha}(B(0)))) = (z_0, \tilde{\psi}(h_{\alpha}(\alpha))) = (z_0, \tilde{\psi}(0)) = (z_0, \psi(0)) = (z_0, w_0)$. Moreover, $\chi(\mu_j) = (\varphi(\mu_j), \tilde{\psi}(h_{\alpha}(B(\mu_j)))) = (a_j, \tilde{\psi}(h_{\alpha}(0))) = (a_j, \tilde{\psi}(0))$.
If \( \beta \) Let \( \alpha \)

Recently, Theorem 1.12.14 has been extended in the following way

Moreover, \( \mu \) and \( \alpha \)

\[ \delta_{G \times D}(A \times \{b\}, (z_0, w_0)) \leq \prod_{j=1}^{N} |\mu_j| < \delta_{G}(A, z_0) + \varepsilon. \]

\[ (b) \ \delta_{G}(A, z_0) < \delta_{D}(b, w_0). \]

Take \( \eta \in E \) and \( \psi \in \mathcal{O}(E, D) \) such that \( \psi(0) = w_0, \psi(\eta) = b, \) and \( |\eta| < \delta_{D}(b, w_0) + \varepsilon/2. \)

Take \( \mu_1, \ldots, \mu_N \in E \) and \( \varphi \in \mathcal{O}(E, G) \) such that \( \varphi(0) = z_0, \varphi(\mu_j) = a_j, \ j = 1, \ldots, N, \) and \( \prod_{j=1}^{N} |\mu_j| < \delta_{D}(b, w_0) + \varepsilon/2. \) Using the same argument as in (a), we may assume that \( \mu_j \neq 0, \ j = 1, \ldots, N. \)

**Lemma 1.12.15.** Let \( \mu_1, \ldots, \mu_N \in E_+, \alpha := \prod_{j=1}^{N} |\mu_j|. \) Assume that \( \alpha < \beta < \alpha^{1/2}, \) with \( k \in \mathbb{N}. \) For \( t \in [0, 1], \) put \( f_t(z) := z \left( \frac{\alpha - 1}{\alpha - 1 + t^2} \right)^k, \) \( z \in E. \) Then there exist \( t \in [0, 1] \) and \( \mu_j \in E \) such that \( f_t(\mu_j) = \mu_j, \) \( j = 1, \ldots, N, \) and \( \prod_{j=1}^{N} |\mu_j| = \beta. \)

**Proof.** (Due to W. Zwonek (51)) Observe that \( f_1(E) = E. \) Hence, for any \( j \in \{1, \ldots, N\} \) there exists a \( \mu_j \in E \) such that \( f_t(\mu_j) = \mu_j. \) Let \( \Phi_j(t) := \min\{|\mu_j| : f_t(\mu_j) = \mu_j\}, \) \( \Phi := \Phi_1 \cdots \Phi_N. \) We have \( \Phi(0) = \alpha^{1/2} > \beta, \) \( \Phi(1) = \alpha < \beta. \) We only need to show that each function \( \Phi_j \) is continuous.

Fix a \( j \in \{1, \ldots, N\} \) and let \( \Phi_j(t) = |\mu_j(t)|, \ t \in [0, 1]. \) Suppose that \( [0, 1] \ni t_s \to t_0 \) and \( |\mu_j(t_s)| \leq m < \Phi_j(t_0). \) Then, without loss of generality, \( \mu_j(t) \to \mu_j \in E. \) Therefore, \( f_t(\mu_j) = \mu_j, \) i.e. \( \Phi_j(t_0) \leq m; \) contradiction. So, \( \Phi_j \) is lower semicontinuous.

On the other hand, by the Hurwitz theorem, for any \( \varepsilon > 0, \) the equation \( f_t(z) = \mu_j \) must have a solution in the disc \( \mathbb{D}(\mu_j(t_0), \varepsilon), \) provided \( s \gg 1. \) Hence \( \Phi_j(t_s) < \Phi_j(t_0) + \varepsilon, \) \( s \gg 1, \) and finally \( \lim_{s \to +\infty} \Phi_j(t_s) \leq \Phi_j(t_0). \) \( \square \)

Using Lemma 1.12.15 with \( \beta := \delta_{D}(b, w_0) + \varepsilon/2, \) we may modify \( \varphi \) and \( \mu_1, \ldots, \mu_N \) in such a way that \( |\eta| < \prod_{j=1}^{N} |\mu_j| < \delta_{D}(b, w_0) + \varepsilon. \) Now we continue as in (a). \( \square \)

**Remark 1.12.16.** Lemma 1.12.15 was improved by N. Nikolov, namely:

Let \( \mu_1, \ldots, \mu_N \in E_+, \alpha := \prod_{j=1}^{N} |\mu_j|, \alpha < \beta < 1. \) For \( t \in [0, 1], \) put \( f_t(z) := z \left( \frac{\alpha - 1}{\alpha - 1 + t^2} \right)^k, \) \( z \in E. \) Then there exist \( t \in [0, 1] \) and \( \mu_j, \ldots, \mu_N \in E \) such that \( f_t(\mu_j) = \mu_j, \ j = 1, \ldots, N, \) and \( \prod_{j=1}^{N} |\mu_j| = \beta. \)

Indeed, the case where \( \beta < \alpha^{1/2} \) reduces to the proof of Lemma 1.12.15 (with \( k = 1). \)

If \( \beta \geq \alpha^{1/2}, \) then put \( \Psi_j(t) := \max\{|\mu_j| : f_t(\mu_j) = \mu_j\}, \) \( \Psi := \Psi_1 \cdots \Psi_N. \) Observe that \( \Psi(0) = \alpha^{1/2}. \) Similarly as in Lemma 1.12.15 we prove that \( \Psi_j \) is continuous on \([0, 1].\)

Moreover, \( \Psi_j(t) \to 1 \) when \( t \to 1, \ j = 1, \ldots, N. \)

**Remark 1.12.17.** Recently, Theorem 1.12.14 has been extended in the following way in [Nik-Zwo 2004].

**Theorem.** Let \( D \subset \mathbb{C}^n \) and \( G \subset \mathbb{C}^m \) be domains and let \( z \in D, \ w \in G, \ A \subset D. \) Then

\[ \max\{\delta_D(A, z), \ i_G^A(b, w)\} \leq \delta_{D \times G}(A \times \{b\}, (z, w)) \leq \max\{\delta_D(A, z), \ \delta_G(b, w)\}, \]

where
\[
l_N^G(b, w) := \inf \left\{ \prod_{j=1}^{N} |\lambda_j| : (\lambda_j)_{j=1}^{N} \subset E, \exists \varphi \in C(E, G) : \right. \\
\left. \varphi(0) = w, \varphi(\lambda_j) = b, \#\{k : \lambda_k = \lambda_j\} \leq \operatorname{ord}_{\lambda_j}(\varphi - b), j = 1, \ldots, N \right\}, \quad N \in \mathbb{N},
\]
\[
l_\infty^G(b, w) := \inf \left\{ \prod_{j=1}^{\infty} |\lambda_j| : (\lambda_j)_{j=1}^{\infty} \subset E, \exists \varphi \in C(E, G) : \right. \\
\left. \varphi(0) = w, \varphi(\lambda_j) = b, \#\{k : \lambda_k = \lambda_j\} \leq \operatorname{ord}_{\lambda_j}(\varphi - b), j = 1, 2, \ldots, \right\}.
\]
Moreover, for any \( N \in \mathbb{N} \cup \{\infty\} \) the equality
\[
\delta_{D \times G}(A \times \{b\}, (z, w)) = \max\{\delta_D(A, z), \delta_G(b, w)\}
\]
holds for any \( A \subset D \) with \#\( A = N \) if and only if \( \delta_G(b, w) = l_N^G(b, w) \).

Moreover, they proved the following result.

**Theorem.** Let \( A, B \subset E, \#A = \#B = 2 \), and \( z, w \in E \) be such that \( \delta_E(A, z) = \delta_E(B, w) \). Then
\[
\delta_E(A, z) = \min\{\delta_{E^2}(C, (z, w)) : C \subset A \times B\}
\]
if and only if there is an \( h \in \operatorname{Aut}(E) \) with \( h(z) = w \) and \( h(A) = B \).

Consequently, if \( \zeta \in E \setminus A \), then there exist uncountably many \( \xi \in E \) for which
\[
\delta_E(A, \zeta) = \delta_E(B, \xi) < \min\{\delta_{E^2}(C, (\zeta, \xi)) : C \subset A \times B\} \leq \delta_{E^2}(A \times B, (\zeta, \xi))
\]
and, therefore,
\[
g_{E^2}(A \times B, (\zeta, \xi)) < \delta_{E^2}(A \times B, (\zeta, \xi)),
\]
cf. Example 1.11.4.

**1.12.3. Product property for \( d_G^{\text{min}} \) and \( d_G^{\text{max}} \).** The case of generalized Green function suggests that the product property might hold for other generalized holomorphically contractible families \((d_G)\), i.e.

\[
d_{G \times D}(A \times B, (z, w)) = \max\{d_G(A, z), d_D(B, w)\}, \quad (z, w) \in G \times D,
\]
(P)

for any domains \( G \subset \mathbb{C}^n, D \subset \mathbb{C}^m \) and for any sets \( \emptyset \neq A \subset G, \emptyset \neq B \subset D \). Notice that the inequality “\( \geq \)" follows from (H) applied to the projections \( G \times D \rightarrow G, G \times D \rightarrow D \).

The definition applies to the standard holomorphically contractible families and means that
\[
d_{G \times D}((a, b), (z, w)) = \max\{d_G(a, z), d_D(b, w)\}, \quad (a, b), (z, w) \in G \times D.
\]

Recall that the standard (non generalized) families \((h_G^*)\), \((c_G^*)\), \((g_G)\) have the product property; cf. [J-P 1993], Ch. 9; see also [Mey 1997] (for a proof of the product property for the Möbius functions based on functional analysis methods) and [Jar-Pfl 1999b] (for the case of complex spaces). Moreover, it is known that the higher order Möbius functions \((m_G^{(k)})\) with \( k \geq 2 \) have no product property; cf. [J-P 1993], Ch. 9.

1.12. Product property
Proposition 1.12.18. The system \( (d_G^{\max}) \) has the product property.

Proof. Fix \((z_0, w_0) \in G \times D \) and \( \varepsilon > 0 \). Let \((a, b) \in A \times B \) be such that \( \tilde{k}^*_G(a, z_0) \leq d_G^{\max}(A, z_0) + \varepsilon \), \( \tilde{k}^*_D(b, w_0) \leq d_D^{\max}(B, w_0) + \varepsilon \). Then using the product property for \((\tilde{k}^*_G)_G\), we get

\[
d_G^{\max}(A \times B, (z_0, w_0)) \leq \tilde{k}^*_G(A, z_0) + \tilde{k}^*_D(B, w_0) = \max\{\tilde{k}^*_G(a, z_0), \tilde{k}^*_D(b, w_0)\} \leq \max\{d_G^{\max}(A, z_0), d_D^{\max}(B, w_0)\} + \varepsilon.
\]

We do not know whether the system \((d_G^{\min}) \) has the product property. So far we were able to manage \([\text{Jar-Jar-Pf 2003}] \) only the case where \#B = 1 — see Proposition 1.12.20 (see also \([\text{Die-Tra 2003}] \)). Recall that \(d_G^{\min}(A, \cdot) = m_G(A, \cdot) \) — Proposition 1.5.4.

Proposition 1.12.19. Assume that for any \( n \in \mathbb{N} \), the system \((m_G)_G\) has the following special product property:

\[
|\Psi(z, w)| \leq (\max_{G \times D}|\Psi|) \max\{m_G(A, z), m_D(B, w)\}, \quad (z, w) \in G \times D,
\]

where \( G, D \subset \mathbb{C}^n \) are balls with respect to arbitrary \( \mathbb{C} \)-norms, \( A \subset D, B \subset G \) are finite and non-empty, \( \Psi(z, w) := \sum_{j=1}^n z_j w_j \), and \( \Psi|_{A \times B} = 0 \). Then the system \((m)_G\) has the product property \((P)\) in the full generality.

Moreover, if \((P)\) holds with \#B = 1, then \((P)\) holds with \#B = 1.

Proof. (Cf. \([\text{J-P 1993}, \text{the proof of Theorem 9.5.}] \)) Fix arbitrary domains \( G \subset \mathbb{C}^n \), \( D \subset \mathbb{C}^m \), non-empty sets \( A \subset G, B \subset G \), and \((z_0, w_0) \in G \times D \). We have to prove that for any \( F \in \mathcal{O}(G \times D, E) \) with \( F|_{A \times B} = 0 \) the following inequality is true:

\[
|F(z_0, w_0)| \leq \max\{m_G(A, z_0), m_D(B, w_0)\}.
\]

By Remark 1.6.1(h), we may assume that \( A, B \) are finite.

Let \((G_\nu)_{\nu=1}^\infty, (D_\nu)_{\nu=1}^\infty \) be sequences of relatively compact subdomains of \( G \) and \( D \), respectively, such that \( A \cup \{z_0\} \subset G_\nu \setminus G, B \cup \{w_0\} \subset D_\nu \setminus D \). By Remark 1.6.1(h), it suffices to show that

\[
|F(z_0, w_0)| \leq \max\{m_{G_\nu}(A, z_0), m_{D_\nu}(B, w_0)\}, \quad \nu \geq 1.
\]

Fix a \( \nu_0 \in \mathbb{N} \) and let \( G' := G_{\nu_0}, D' := D_{\nu_0} \).

It is well known that \( F \) may be approximated locally uniformly in \( G \times D \) by functions of the form

\[
F_s(z, w) = \sum_{\mu=1}^{N_s} f_{s, \mu}(z) g_{s, \mu}(w), \quad (z, w) \in G \times D,
\]

where \( f_{s, \mu} \in \mathcal{O}(G), g_{s, \mu} \in \mathcal{O}(D), s \geq 1, \mu = 1, \ldots, N_s \). Notice that \( F_s \longrightarrow 0 \) uniformly on \( A \times B \). Using Lagrange interpolation formula, we find polynomials \( P_s : \mathbb{C}^n \times \mathbb{C}^m \longrightarrow \mathbb{C} \) such that \( P_s|_{A \times B} = F_s|_{A \times B} \) and \( P_s \longrightarrow 0 \) locally uniformly in \( \mathbb{C}^n \times \mathbb{C}^m \). The functions \( \tilde{F}_s := F_s - P_s, s \geq 1 \), also have form \((1.12.26)\) and \( \tilde{F}_s \longrightarrow F \) locally uniformly in \( G \times D \). Hence, without loss of generality, we may assume that \( F_s|_{A \times B} = 0, s \geq 1 \). Let
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$m_s := \max\{1, \|F_s\|_{G' \times D'}\}$ and \(\tilde{F}_s := F_s/m_s, \ s \geq 1\). Note that \(m_s \to 1\) and, therefore, \(\tilde{F}_s \to F\) uniformly on \(G' \times D'\). Consequently, we may assume that \(F_s(G' \times D') \subset E, \ s \geq 1\).

It is enough to prove that

\[|F_s(z_0, w_0)| \leq \max\{m_G(A, z_0), \ m_D(B, w_0)\}, \ s \geq 1 .\]

Fix an \(s = s_0 \in \mathbb{N}\) and let \(N := N_{s_0}, \ f_\mu := f_{s_0, \mu}, \ g_\mu := g_{s_0, \mu}, \ \mu = 1, \ldots, N\). Let \(f := (f_1, \ldots, f_N) : G \to \mathbb{C}^N\) and \(g := (g_1, \ldots, g_N) : D \to \mathbb{C}^N\). Put

\[K := \{\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N : \ |\xi_\mu| \leq \|f_\mu\|_{G'}, \ \mu = 1, \ldots, N, \ |\Psi(\xi, g(w))| \leq 1, \ w \in D'\} .\]

It is clear that \(K\) is an absolutely convex compact subset of \(\mathbb{C}^N\) with \(f(G') \subset K\). Let

\[L := \{\eta = (\eta_1, \ldots, \eta_N) \in \mathbb{C}^N : \ |\eta_\mu| \leq \|g_\mu\|_{D'}, \ \mu = 1, \ldots, N, \ |\Psi(\xi, \eta)| \leq 1, \ \xi \in K\} .\]

Then again \(L\) is an absolutely convex compact subset of \(\mathbb{C}^N\), and moreover, \(g(D') \subset L\).

Let \((W_\sigma)_{\sigma=1}^{\infty}\) (resp. \((V_\sigma)_{\sigma=1}^{\infty}\)) be a sequence of absolutely convex bounded domains in \(\mathbb{C}^N\) such that \(W_{\sigma+1} \subset W_\sigma\) and \(W_\sigma \setminus K\) (resp. \(V_{\sigma+1} \subset V_\sigma\) and \(V_\sigma \setminus L\)). Put \(M_\sigma := \|\Psi\|_{W_\sigma \times V_\sigma}, \ \sigma \in \mathbb{N}\). By (P_0) and by the holomorphic contractibility applied to the mappings \(f : G' \to W_\sigma, \ g : D' \to V_\sigma, \ \sigma > 0\) and \(g(D') \subset L\), we have

\[|F_{s_0}(z_0, w_0)| = \|\Psi(f(z_0), g(w_0))\| \\ \leq M_\sigma \max\{m_{W_\sigma}(f(A), f(z_0)), \ m_{V_\sigma}(g(B), g(w_0))\} \\ \leq M_\sigma \max\{m_G(A, f(z_0)), \ m_D(g(B), w_0)\} \leq M_\sigma \max\{m_G(A, z_0), \ m_D(B, w_0)\} .\]

Letting \(\sigma \to +\infty\) we get the required result. \(\square\)

**Proposition 1.12.20.** The system \((m_G)_G\) has the product property (P) whenever \#B = 1, i.e. for any domains \(G \subset \mathbb{C}^n, \ D \subset \mathbb{C}^m\), for any set \(A \subset G\), and for any point \(b \in D\) we have

\[m_{G \times D}(A \times \{b\}, (z, w)) = \max\{m_G(A, z), \ m_D(b, w)\}, \ (z, w) \in G \times D .\]

**Proof.** By Proposition 1.12.19, we only need to check (P) in the case, where \(D\) is a bounded convex domain, \(A\) is finite, and \(B = \{b\}\). Fix \((z_0, w_0) \in G \times D\). Let \(\varphi : E \to D\) be a holomorphic mapping such that \(\varphi(0) = b\) and \(\varphi(m_D(b, w_0)) = w_0\) (cf. [J-P 1993], Ch. 8). Consider the mapping \(F : G \times E \to G \times D, \ F(z, \lambda) := (z, \varphi(\lambda))\). Then

\[m_{G \times D}(A \times \{b\}, (z_0, w_0)) \leq m_{G \times E}(A \times \{0\}, (z_0, m_D(b, w_0))) .\]

Consequently, it suffices to show that

\[m_{G \times E}(A \times \{0\}, (z_0, \lambda)) \leq \max\{m_G(A, z_0), \ |\lambda|\}, \ \lambda \in E .\]  \hspace{1cm} (1.12.27)

The case where \(m_G(A, z_0) = 0\) is elementary: for an \(f \in \mathcal{O}(G \times E, E)\) with \(f|_{A \times \{0\}} = 0\) we have \(f(z_0, 0) = 0\) and hence \(|f(z_0, \lambda)| \leq |\lambda|, \ \lambda \in E\) (by the Schwarz lemma). Thus, we may assume that \(r := m_G(A, z_0) > 0\). First observe that it suffices to prove (1.12.27) only on the circle \(|\lambda| = r\). Indeed, if the inequality holds on that circle, then by the maximum principle for subharmonic functions (applied to the function \(m_{G \times E}(A \times \{0\}, (z_0, \cdot))\) it
holds for all $|\lambda| \leq r$. In the annulus $\{r < |\lambda| < 1\}$ we apply the maximum principle to the subharmonic function $\lambda \mapsto \frac{1}{|\lambda|} m_{G \times E}(A \times \{0\}, (z_0, \lambda))$.

Now fix a $\lambda_0 \in E$ with $|\lambda_0| = r$. Let $f$ be an extremal function for $m_{G}(A, z_0)$ with $f|_A = 0$ and $f(z_0) = \lambda_0$. Consider $F : G \rightarrow G \times E$, $F(z) := (z, f(z))$. Then

$$m_{G}(A \times \{0\}, (z_0, \lambda_0)) \leq m_{G}(A, z_0) = \max \{m_{G}(A, z_0), |\lambda_0|\},$$

which completes the proof. □
CHAPTER 2

Hyperbolicity and completeness

2.1. $c^i$-hyperbolicity versus $c$-hyperbolicity

Recall that a domain $G \subset \mathbb{C}^n$ is called $c_G^i$-hyperbolic (or shortly $c^i$-hyperbolic), respectively $c_G$-hyperbolic (shortly $c$-hyperbolic), if $c_G^i$, respectively $c_G$, is a true distance on $G$. In virtue of the inequality $c_G \leq c_G^i$, if $G$ is $c$-hyperbolic, then it is $c_G^i$-hyperbolic. If $G$ is bounded, then $c_G$ is a distance. In the general case, the following result due to J.-P. Vigué (cf. [Vig 1996]) gives a characterization of $c_G^i$-hyperbolicity.

**Theorem 2.1.1.** Let $G \subset \mathbb{C}^n$ be a domain. Then the following properties are equivalent:

(i) $G$ is $c_G^i$-hyperbolic;

(ii) there is no non-constant $C^1$-curve $\alpha : [0, 1] \rightarrow G$ such that $\gamma_G(\alpha; \alpha') \equiv 0$;

(iii) for any point $a \in G$ there exists a neighborhood $U = U(a) \subset G$ such that $c_G(a, z) \neq 0$, $z \in U \setminus \{a\}$.

**Proof.** (i) $\Rightarrow$ (ii): Suppose the contrary, namely, that there exists a $C^1$-curve $\alpha : [0, 1] \rightarrow G$ such that

$$\gamma_G(\alpha; \alpha') \equiv 0, \quad \alpha'(t_0) \neq 0 \quad \text{for a } t_0 \in [0, 1].$$

Obviously, then for any $0 \leq t' < t'' \leq 1$ we have $c_G^i(\alpha(t'), \alpha(t'')) = 0$. In virtue of $\alpha'(t_0) \neq 0$ there are two different points $\alpha(t')$, $\alpha(t'')$ showing that $G$ is not $c^i$-hyperbolic. Contradiction.

(ii) $\Rightarrow$ (iii): We proceed by assuming the contrary. So let $a \in G$ be such a point that there exists a sequence of points $(z^j)_{j \in \mathbb{N}} \subset G \setminus \{a\}$, $z^j \rightarrow a$, such that $c_G(a, z^j) = 0$, $j \in \mathbb{N}$. We have to find a $C^1$-curve which does fulfill the property stated in (ii).

Observe that $A := \{ z \in G : c_G(a, z) = 0 \} = \{ z \in G : f(a) = f(z), f \in O(G, E) \}$ is an analytic subset of $G$. In virtue of the existence of the points $z^j \in A \setminus \{a\}$ tending to $a$, the dimension of the analytic set $A$ in $a$ is at least 1. Therefore, there is a $C^1$-curve $\alpha : [0, 1] \rightarrow \text{Reg} A$ such that $\alpha' \neq 0$. On the other side, since this curve lies in $A$, we have $\gamma_G(\alpha; \alpha') \equiv 0$: contradiction.

(iii) $\Rightarrow$ (i): Fix $a, b \in G$, $a \neq b$, and choose a neighborhood $U = U(a) \subset G$ according to (iii). Moreover, let $V = V(a) \Subset U$, $b \notin V$. Obviously, $0 < c_G(a, z)$, $z \in U \setminus \{a\}$. Applying the continuity of $c_G$ there is a $C > 0$ such that $c_G(a, \cdot)|_{\partial V} \geq C$. Thus for any $C^1$-curve $\alpha : [0, 1] \rightarrow G$, $\alpha(0) = a$, $\alpha(1) = b$, there is a $t_0 \in (0, 1)$ with $\alpha(t_0) \in \partial V$; therefore, $L_\alpha(\alpha) \geq c_G(a, \alpha(t_0)) + c_G(\alpha(t_0), b) \geq C > 0$. Hence, $c_G^i(a, b) \geq C > 0$. □

Moreover, there is the following general relation between $\gamma_G$-hyperbolicity and local $c$-hyperbolicity.
Proposition 2.1.2. Any domain \( G \subset \mathbb{C}^n \) that is \( \gamma_G \)-hyperbolic (i.e. \( \gamma_G(z; X) > 0, \ z \in G, \ X \in \mathbb{C}^n \setminus \{0\} \)) is locally \( c_G \)-hyperbolic (i.e. for any \( a \in G \) there exists a neighborhood \( U = U(a) \subset G \) such that \( c_G \) is a distance on \( U \)). In particular, \( G \) is \( c^i \)-hyperbolic.

Proof. Fix an \( a \in G \) and suppose that \( z_j, w_j \to a, \ z_j \neq w_j, \ c_G(z_j, w_j) = 0, \ j = 1, 2, \ldots \). We may assume that \( \frac{z_j - w_j}{\|z_j - w_j\|} \to X_0 \in \partial B_n \). Then
\[
\gamma_G(a; X_0) = \lim_{j \to \infty} \frac{c_G(z_j, w_j)}{\|z_j - w_j\|} = 0;
\]
contradiction (cf. §1.2). \( \square \)

Observe that the result is true for any \( C^1 \)-pseudodistance (cf. §1.2.4).

Remark 2.1.3. It seems to be unknown whether \( c \)-hyperbolicity implies \( \gamma \)-hyperbolicity?

Example 2.1.4. There is a domain \( G \subset \mathbb{C}^3 \) which is not \( c_G \)-hyperbolic and not \( \gamma_G \)-hyperbolic, but nevertheless \( c_i \)-hyperbolic (see [Vig 1996]). This \( G \) is constructed via an example of a 1-dimensional complex space and then applying the Remmert embedding theorem. We omit here details.

Remark 2.1.5. Notice that the Example 2.1.4 is not explicitly given. So it is interesting to find an effective example of that type; moreover, the question whether such an example is possible in \( \mathbb{C}^2 \) is still an open one.

2.2. Hyperbolicity for Reinhardt domains

Before we shall discuss the different notions of hyperbolicity in the case of pseudoconvex Reinhardt domains, recall the effective formulas for the Kobayashi pseudodistance on elementary Reinhardt domains (cf. Theorem 1.3.1). Let
\[
V_j := \{ z \in \mathbb{C}^n : z_j = 0 \}, \ j = 1, \ldots, n.
\]
Moreover, for a matrix \( A = (A_{jk})_{j=1,\ldots,n, \ k=1,\ldots,n} \in \mathbb{Z}^{n \times n} \), we denote by \( A^j \) its \( j \)-th row. Put
\[
\Phi_A : \mathbb{C}^n \to \mathbb{C}^n, \ \Phi(z) := (z^{A_1}, \ldots, z^{A_n}).
\]

Theorem 2.2.1 ([Zwo 1999a]). Let \( G \) be a pseudoconvex Reinhardt domain in \( \mathbb{C}^n \). Then the following properties are equivalent:

(i) \( G \) is \( c_G \)-hyperbolic;

(ii) \( G \) is \( \tilde{k}_G \)-hyperbolic;

(iii) \( G \) is Brody-hyperbolic (i.e. \( \mathcal{O}(\mathbb{C}, G) = \mathbb{C} \));

(iv) \( \log G \) contains no affine lines, and either \( V_j \cap G = \emptyset \) or \( V_j \cap G \) is \( c \)-hyperbolic as a domain in \( \mathbb{C}^{n-1}, \ j = 1, \ldots, n; \)

\[\text{there exist } A = (A_{jk})_{j=1,\ldots,n, k=1,\ldots,n} \in \mathbb{Z}^{n \times n}, \ \text{rank } A = n, \ \text{and a vector} \]
\[C = (C_1, \ldots, C_n) \in \mathbb{R}^n \text{ such that} \]
\[\log G := \{ x \in \mathbb{R}^n : (e^{x_1}, \ldots, e^{x_n}) \in G \}.\]
\[ G \subseteq G(A, C) := D_{A_1, C_1} \cap \cdots \cap D_{A_n, C_n}, \]

- \( G \subseteq G(A, C) \) (cf. (iv)),
- \( \det G(A, C) \) (cf. (iv)).

(iv’) there exist \( A \in \mathbb{Z}(n \times n) \), \( |\det A| = 1 \), and a vector \( C \in \mathbb{R}^n \) such that

\[ G \subseteq G(A, C), \]

- either \( V_j \cap G = \emptyset \) or \( V_j \cap G \) is \( c \)-hyperbolic as a domain in \( \mathbb{C}^{n-1} \), \( j = 1, \ldots, n; \)

(v) \( G \) is algebraically equivalent to a bounded domain (i.e. there is a matrix \( A \in \mathbb{Z}(n \times n) \) such that \( \Phi_A \) is defined on \( G \) and gives a biholomorphic mapping from \( G \) to the bounded domain \( \Phi_A(G) \)).

(vi) \( G \) is \( k \)-complete.

In the sequel a domain of the type \( G(A, C) \) (cf. (iv) in Theorem 2.2.1) will be shortly called a \textit{quasi-elementary} Reinhardt domain.

To prove Theorem 2.2.1 we need the following lemmas.

\textbf{Lemma 2.2.2 ([Zwo 1999a]).} Let \( G(A, C) \) be as in Theorem 2.2.1. Then:

(a) there is a matrix \( \tilde{A} \in \mathbb{Z}(n \times n) \), \( |\det \tilde{A}| = 1 \), and a vector \( \tilde{C} \in \mathbb{R}^n \) such that \( G(A, C) \subseteq G(\tilde{A}, \tilde{C}) \);

(b) \( c_{G(A,C)}(z, w) > 0 \) for any points \( z, w \in G(A, C) \cap \mathbb{C}^n \), \( z \neq w \).

\textbf{Proof.} Fix a matrix \( A \) and a vector \( C \) as in Lemma 2.2.2.

Step 1. To prove (a) it suffices to construct a sequence of quasi-elementary Reinhardt domains \( G_0 := G(A, C) \subset \cdots \subset G_N \) such that \( |\det G_j| < |\det G_{j+1}| \), where \( \det G(A, C) := \det A \).

Assume that \( G_j \) has been already constructed. Let \( G_j = G(B, D) \) with a matrix \( B \in \mathbb{Z}(n \times n) \), \( |\det B| \geq 1 \), and a vector \( D \in \mathbb{R}^n \). In case when \( |\det B| > 1 \) we describe how to get \( G_{j+1} \).

Put

\[ S(G_j) := \{ \alpha \in \mathbb{Z}^n : z^\alpha \in \mathcal{H}^\infty (G_j) \}, \]

\[ B(G_j) := S(G_j) \setminus (S(G_j) + S(G_j)). \]

It is known (cf. [J-P 1993], Lemma 2.7.6) that

\[ S := S(G(B, D)) = \mathbb{Z}^n \cap (\mathbb{Q}_+ B^1 \cup \cdots \cup \mathbb{Q}_+ B^n), \]

\[ B := B(G(B, D)) \subset \mathbb{Z}^n \cap (\mathbb{Q} \cap [0, 1) B^1 \cup \cdots \cup \mathbb{Q} \cap [0, 1) B^n) \cup \{ B^1, \ldots, B^n \}. \]

Claim: \( B \not\subset \{ B^1, \ldots, B^n \} \).

Assume the contrary, i.e. \( B \subset \{ B^1, \ldots, B^n \} \). Define

\[ r(B) := \min \{ r \in \mathbb{N} : \text{ if } x \in \mathbb{Q}^n, \text{ } xB \in \mathbb{Z}^n \}, \text{ then } \exists x \in \mathbb{Z}^n \}. \]

Observe that \( B^{-1} B \in \mathbb{Z}(n \times n) \), i.e. all the rows of \( B^{-1} \) are special vectors in the definition of the number \( r(B) \). So \( r(B)B^{-1} \in \mathbb{Z}(n \times n) \), from which \( r(B)^n = \det(r(B)B^{-1}B) = \det(r(B)B^{-1}) \det(B) \) follows. Therefore, if \( r(B) = 1 \) then \( |\det B| = 1 \), which gives the contradiction.

So it remains to prove that \( r(B) = 1 \).

Take an arbitrary \( x \in \mathbb{Q}^n \) with \( xB \in \mathbb{Z}^n \). We have to show that \( x \in \mathbb{Z}^n \). In fact: we write \( xB = uB + \nu B \), where \( u = (u_1, \ldots, u_n), u_j := x_j - [x_j] \geq 0 \) and \( \nu = (\nu_1, \ldots, \nu_n), \nu_j := [x_j] \in \mathbb{Z}, j = 1, \ldots, n \) (here \( [x] \) denotes the largest integer smaller or equal \( x \)). Obviously, \( uB \in \mathbb{Z}^n \). Applying the above description of \( S \), it follows that
particular, the mapping $\Phi$ is proper iff $\det \tilde{B}$ is zero. Then
with $\tilde{B}$ that matrix whose rows $B^j$ are given by $B^j := \beta$, $\beta \in \mathbb{B}$, $j = 2, \ldots, n$. Moreover, with $\tilde{C}_1 := \sum_{j=1}^n t_j C_j$ and $\tilde{C}_j := C_j$, $j = 2, \ldots, n$, we put

$$G_{j+1} := G(\tilde{B}, \tilde{C}),$$

where $\tilde{C} := (\tilde{C}_1, \ldots, \tilde{C}_n)$.

Then $\det \tilde{B} = t_1 \det B < \det B$ and $G_j \subset G_{j+1}$. Hence, (a) is verified.

**Step 2.** Recall that for a matrix $A \in \mathbb{Z}(n \times n)$ the mapping

$$\Phi_A : \mathbb{C}_n^* \to \mathbb{C}_n^*, \quad \Phi_A(z) := (z^{A_1}, \ldots, z^{A_n}),$$

is proper iff $\det A \neq 0$, and that in this case its multiplicity is given by $|\det A|$. In particular, the mapping $\Phi_{\tilde{A}}$ of (a), is a biholomorphic mapping from $\mathbb{C}_n^*$ to itself.

Now fix two different points $z, w \in G(A, C) \cap \mathbb{C}_n^*$. Then

$$c_{G(A,C)}(z, w) \geq c_{G(\tilde{A}, \tilde{C})}(z, w) = c_{G(\tilde{A}, \tilde{C})} \cap \mathbb{C}_n^*(z, w) = c_{E_n}(\Psi(z), \Psi(w)) > 0,$$

where $\Psi(z) := (\tilde{\Phi}_1(z)/\tilde{\epsilon}_{C_1}, \ldots, \tilde{\Phi}_n(z)/\tilde{\epsilon}_{C_n})$ with $\tilde{\Phi}_A := (\Phi_1, \ldots, \Phi_n)$. □

**Lemma 2.2.3.** Let $\Omega \subset \mathbb{R}^n$ be a convex domain containing no straight lines. Then there are linearly independent vectors $A^1, \ldots, A^n \in \mathbb{Z}^n$ and a $C \in \mathbb{R}^n$ such that

$$\Omega \subset \{x \in \mathbb{R}^n : \langle x, A^j \rangle < C_j, \ j = 1, \ldots, n\}.$$

**Proof.** Cf. [Vla 1993]. □

**Proof of Theorem 2.2.1.** First, observe that the implications (i) $\implies$ (ii) $\implies$ (iii) are obvious and that (iv) $\implies$ (iv') is true due to Lemma 2.2.2.

The remaining proof uses induction on the dimension $n$. Obviously, the theorem is true in the case $n = 1$. Now, let $n \geq 2$.

(iii) $\implies$ (iii'): The first condition is an obvious consequence of (iii). The second one follows from the induction process.

(iii') $\implies$ (iv): Note that the second condition in (iv) follows from applying the theorem in the case $n - 1$. From (iii) we see that $\log G$ does not contain straight lines. Therefore, we immediately get (iv) from Lemma 2.2.3.

(iv') $\implies$ (i): Take $z, w \in G, z \neq w$. Case 1: If both points belong to $\mathbb{C}_n^*$, then, in virtue of Lemma 2.2.2, we have

$$c_G(z, w) \geq c_{G(A,C)}(z, w) > 0.$$

Case 2: Let $z \in \mathbb{C}_n^*, w \notin \mathbb{C}_n^*$. Without loss of generality we may assume that $w = (w_1, \ldots, w_k, 0, \ldots, 0)$ with $w_1 \cdots w_k \neq 0$. Then $k < n$ and $A^j_s \geq 0, j = 1, \ldots, n, s = k + 1, \ldots, n$. Since $\text{rank } A = n$ we find a $j \in \{1, \ldots, n\}$ and an $r \in \{k + 1, \ldots, n\}$ such that $A^j_r > 0$. Thus $w^{A_j} = 0 \neq z^{A^j}$. Therefore,

$$c_G(z, w) \geq c_{G(A^j,C_j)}(z, w) \geq c_D(z^{A^j}, w^{A^j}) > 0,$$

where $D := e^{C_j} E$. 

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Case 3: Let \( z, w \not\in \mathbb{C}^n_1 \). We may assume that \( z_1 = 0 \) and \( z_2 \neq w_2 \). Consequently, \( \pi_{2,\ldots,n}(G) \) is \( c \)-hyperbolic and \( \pi_{2,\ldots,n}(z) \neq \pi_{2,\ldots,n}(w) \). Therefore,
\[
c_G(z, w) \geq c_{\pi_{2,\ldots,n}(G)}(\pi_{2,\ldots,n}(z), \pi_{2,\ldots,n}(w)) > 0.
\]
Hence \( G \) is \( c \)-hyperbolic.

(iv') \( \implies \) (v): By (iv') we know that there is a matrix \( A \in \mathbb{Z}(n \times n), |\det A| = 1 \), and a vector \( \bar{G} \in \mathbb{R}^n \) with \( G \subset G(A, C) \). Moreover, the mapping \( \Phi_A : \mathbb{C}^n_+ \rightarrow \mathbb{C}^n_+ \), \( \Phi_A(z) := (z^{A^1}, \ldots, z^{A^n}) \) is biholomorphic.

Therefore, if the domain \( G \) is contained in \( \mathbb{C}^n_+ \), then \( \Phi_A : G \rightarrow \Phi_A(G) \) is a biholomorphic mapping and \( \Phi_A(G) \) is bounded.

The remaining case is done by induction:
Obviously, the case \( n = 1 \) is clear. So we may assume that \( n \geq 2 \) and, without loss of generality, that \( V_n \cap G \neq \emptyset \).

Claim: It suffices to prove (v) under the additional assumption that
\[
V_n \cap G \neq \emptyset \quad \text{and} \quad \pi_j(G) \text{ is bounded,} \quad j = 1, \ldots, n - 1. \quad (2.2.1)
\]
In fact, put \( \tilde{G} := G \cap V_n \). By assumption, \( \tilde{G} \) is a \( c \)-hyperbolic pseudoconvex Reinhardt domain in \( \mathbb{C}^{n-1} \). By the induction hypothesis there exists a matrix \( \tilde{A} \in \mathbb{Z}((n-1) \times (n-1)) \) such that
\[
\Phi_{\tilde{A}} \text{ is defined on } \tilde{G}, \Phi_{\tilde{A}}(\tilde{G}) \text{ is bounded, and } \Phi_{\tilde{A}} : \tilde{G} \rightarrow \Phi_{\tilde{A}}(\tilde{G}) \text{ is biholomorphic.}
\]

Put
\[
B := \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{Z}(n \times n).
\]

Then \( \Phi_B \) satisfies condition (2.2.1), and so the claim has been verified.

For the remaining part of the proof of (v) we may now assume that (2.2.1) is fulfilled.
Without loss of generality assume that
\[
V_j \cap G \neq \emptyset, \quad j = 1, \ldots, k, \quad V_j \cap G = \emptyset, \quad j = k + 1, \ldots, n - 1.
\]
Put \( \tilde{G} := V_1 \cap \cdots \cap V_k \cap G \). Then \( \tilde{G} \) is a (non empty) \( c \)-hyperbolic pseudoconvex Reinhardt domain. Then there is \( \alpha = (0, \ldots, 0, \alpha_{k+1}, \ldots, \alpha_n) \in S(\tilde{G}), \alpha_n \neq 0 \). The fact that \( \tilde{G} \cap V_n \neq \emptyset \) implies \( \alpha_n > 0 \). Moreover, in virtue of (2.2.1), it is clear that \( e_j := (0, \ldots, 0, 1, 0, \ldots, 0) \in S(\tilde{G}) \) (the number 1 in \( e_j \) is at the \( j \)-th place), \( j = k + 1, \ldots, n - 1 \). Thus
\[
\bar{\alpha} := \frac{1}{\alpha_n} \alpha + \sum_{j=k+1}^{n-1} \left( \frac{\alpha_j}{\alpha_n} \right) + 1 - \frac{\alpha_j}{\alpha_n} e_j \in S(\tilde{G}) \subset S(G).
\]

Define
\[
A := \begin{bmatrix} I_{n-1} & \bar{\alpha}_{k+1} & \cdots & \bar{\alpha}_{n-1} & 0 \\ 0 & \cdots & 0 & \bar{\alpha}_{k+1} & \cdots & \bar{\alpha}_{n-1} & 0 \end{bmatrix}.
\]

Then \( A \) fulfills all the required properties. Hence condition (v) is proved.

(v) \( \implies \) (vi): By assumption we may assume that \( G \) is a bounded pseudoconvex Reinhardt domain. Fix a point \( w \in G \). To verify that \( G \) is \( k \)-complete we only have to
disprove the existence of a sequence \((z^j)_{j \in \mathbb{N}} \subset G\) such that \((k_G(w, z^j))_{j \in \mathbb{N}}\) is bounded, but \(z^j \xrightarrow{j \to \infty} z^0 \in \partial G\).

Case \(z^0 \in \mathbb{C}^2\): We may assume that \(z^0 = (1, \ldots, 1)\). It is clear that there is an \(\alpha \in \mathbb{R}^n, \alpha \neq 0\), such that \(G \subset D_\alpha\), where \(D_\alpha\) denotes the elementary Reinhardt domain for \(\alpha\). Moreover, we may assume that \(\alpha_j \neq 0, j = 1, \ldots, k\), and \(\alpha_{k+1} = \ldots \alpha_n = 0\), where \(k \geq 1\). So we get

\[
    k_G(w, z^j) = k_G(w, z^0) \geq k_{D_\alpha}(w, z^j) = k_{D_\alpha}(\tilde{w}, \tilde{z}^j) = k_{C^n}(\tilde{w}, \tilde{z}^j),
\]

where \(\tilde{\alpha} := (\alpha_1, \ldots, \alpha_k), \tilde{w} := (w_1, \ldots, w_k), \tilde{z}^j := (z_j^1, \ldots, z_j^n)\).

Remark 2.2.4. Observe that Theorem 2.2.1 shows that all notions of hyperbolicity coincide in the class of pseudoconvex Reinhardt domains. That’s why we will often speak only of hyperbolic pseudoconvex Reinhardt domains. Moreover, in that class “hyperbolic” and “Kobayashi-complete” are the same notions.

Remark 2.2.5. The following pseudoconvex Reinhardt domain

\[
    D := \{z \in \mathbb{C}^3 : \max(\{z_1z_2, |z_1z_3|, |z_2|, |z_3|\} < 1)\}
\]

is not \(k\)-hyperbolic since \(\mathbb{C} \times \{0\} \times \{0\} \subset D\); in particular, \(D\) is not \(c\)-hyperbolic.

Let \(\tilde{D} := D \setminus (\mathbb{C} \times \{0\} \times \{0\})\). Then \(\tilde{D}\) is \(c\)-hyperbolic (the functions \(z_1z_2, z_1z_3, z_2, z_3\) separate the points of \(\tilde{D}\)). Observe that \(D\) is the envelope of holomorphy of \(\tilde{D}\), i.e. \(D = \mathcal{H}(\tilde{D})\). Hence, in general, \(c\)-hyperbolicity of a Reinhardt domain and its envelope of holomorphy may be different.

But in the two-dimensional case, there is the following positive result [Die-Hai 2003].

Theorem 2.2.6. Let \(G \subset \mathbb{C}^2\) be a \(c\)-hyperbolic Reinhardt domain. Then its envelope of holomorphy \(\mathcal{H}(G)\) is \(c\)-hyperbolic.
Proof. Recall that the envelope of holomorphy $\mathcal{H}(D)$ of a Reinhardt domain $D \subset \mathbb{C}^n_*$ satisfies the following properties

- $\mathcal{H}(D) \subset \mathbb{C}^n_*$,
- $\log \mathcal{H}(D) = \text{conv}(\log D)$.

Put $G_* := G \cap \mathbb{C}^2_*$. $G_*$ is a Reinhardt domain. Assume that $\log \mathcal{H}(G_*)$ contains an affine line $\ell$. Fix a point $x_0 \in \log G \setminus \ell$. Denote by $\ell'$ the line passing through $x_0$ which is parallel to $\ell$. Then $\ell' \subset \log \mathcal{H}(G_*)$. Let

$$
\ell' = \{(a_1 t + b_1, a_2 t + b_2) : t \in \mathbb{R}\},
$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $a_1^2 + a_2^2 \neq 0$. Hence

$$
A := \{e^{\alpha_0 + b_1}, e^{\alpha_2 + b_2} : \alpha \in \mathbb{C}\} \subset \mathcal{H}(G_*).
$$

Using Liouville’s theorem and the fact that $G$ is $c$–hyperbolic, we get $A \cap G = \emptyset$ or $\ell' \cap \log G = \emptyset$; a contradiction.

Assume now that $\log \mathcal{H}(G)$ contains an affine line. As in the previous step, this leads to a non trivial entire map $\varphi : \mathbb{C} \to \mathcal{H}(G) \cap \mathbb{C}^n_*$. Recall that $\mathcal{H}(G_*) = \mathcal{H}(G) \cap \mathbb{C}^n_*$ (see Theorem 2.5.9 in [Jar-Pfl 2000]). Hence, $\mathcal{H}(G_*)$ contains an affine line; a contradiction.

Without loss of generality, assume finally that $\mathcal{H}(G) \cap V_2 \neq \emptyset$. Denote this intersection by $G' \subset \mathbb{C}$. Suppose that $G'$ is not $c$–hyperbolic. Then either $G' = \mathbb{C}$ or $G' = G_*$. Therefore, either $A_1 := \mathbb{C} \times \{0\} \subset \mathcal{H}(G)$ or $A_2 := \mathcal{H}(G) \cap \mathbb{C}^n_*$ (see Theorem 2.5.9 in [Jar-Pfl 2000]). In virtue of the $c$–hyperbolicity of $G$, we conclude that $A_1 \cap G = \emptyset$ or that $A_2 \cap G = \emptyset$. Therefore, $G \cap V_2 = \emptyset$; a contradiction. Thus Theorem 2.2.1 implies that $\mathcal{H}(G)$ is $c$–hyperbolic. $\square$

We conclude this section with the following result which will be useful later.

**Proposition 2.2.7** ([Zwo 2000a]). Let $G \subset \mathbb{C}^n$ be a hyperbolic pseudoconvex Reinhardt domain. Then the following conditions are equivalent:

(i) $G$ is algebraically equivalent to an unbounded Reinhardt domain;

(ii) $G$ is algebraically equivalent to a bounded Reinhardt domain $D$, for which there is a $j_0$, $1 \leq j_0 \leq n$, such that $\overline{D} \cap V_{j_0} \neq \emptyset$, but $D \cap V_{j_0} = \emptyset$.

**Proof.** (i) $\implies$ (ii): We may assume that $G$ is an unbounded hyperbolic pseudoconvex Reinhardt domain. In virtue of Theorem 2.2.1, there are a bounded Reinhardt domain $D$ and a biholomorphic mapping $\Phi_A : D \to G$ (here we use the notation from Theorem 2.2.1). Suppose that $D$ satisfies the following property:

$$
\text{if } \overline{D} \cap V_{j} \neq \emptyset \text{ then } D \cap V_{j} \neq \emptyset, \quad j = 1, \ldots, n.
$$

Without loss of generality, we may assume that there is a $k \in \{0, 1, \ldots, n\}$ such that

$$
\overline{D} \cap V_{j} \neq \emptyset, \quad j = 1, \ldots, k, \quad D \cap V_{j} = \emptyset, \quad j = k + 1, \ldots, n. \quad (2.2.3)
$$

Now, let $A = \{A_{j}^{r} : r = 1, \ldots, n, j = 1, \ldots, n \in \mathbb{Z}(n \times n)$. Then $A_{j}^{r} \geq 0$, $j = 1, \ldots, k$, $r = 1, \ldots, n$. Moreover, using (2.2.3) and that $D$ is bounded, gives a positive $M$ such that

$$
|z| \geq M, \quad z \in D, \quad k + 1 \leq j \leq n.
$$

Hence, $\sup \{|z|^{M} : z \in D| < \infty, \quad r = 1, \ldots, n$, which implies that $G$ is bounded; contradiction.
2. Hyperbolicity and completeness

(ii) $\Rightarrow$ (i): Observe that the mapping

$$D \ni z \mapsto (z_1, \ldots, z_{j_0-1}, \frac{1}{z_{j_0}}, z_{j_0+1}, \ldots, z_n)$$

maps $D$ biholomorphically onto an unbounded pseudoconvex Reinhardt domain $\hat{D}$. Thus $D$ is algebraically equivalent to $\hat{D}$ and so is $G$.\hfill\Box

2.3. Hyperbolicities for balanced domains

Recall that any balanced domain is $k$–hyperbolic if and only if it is bounded (cf. Theorem 7.1.2 in [Jar-Pfl 1999a]).

Example 2.3.1 ([Azu 1983], see also [J-P 1993], Example 7.1.4). Observe that there is an unbounded pseudoconvex balanced domain $G \subset \mathbb{C}^2$, that is Brody–hyperbolic. To be more concrete $G$ is defined as

$$G := \{z \in \mathbb{C}^2 : h(z) < 1\},$$

where $h(z) := \begin{cases} |z_2|e^{\varphi(z_2)} & \text{if } z_2 \neq 0 \\ |z_1| & \text{if } z_2 = 0 \end{cases}$, and

$$\varphi(\lambda) := \max \left\{ \log|\lambda|, \sum_{j=2}^{\infty} \frac{1}{k^2} \log \left| \lambda - \frac{1}{k} \right| \right\},$$

$\lambda \in \mathbb{C}$. Recently, S.-H. Park [Par 2003] has shown that $G$ is almost $k$–hyperbolic, i.e. $\tilde{k}_G(z, w) > 0$, whenever $z_1 \neq w_1$ or $(z_1 = w_1 \neq 0$ and $z_2 \neq w_2)$.\hfill[7]

It is still unclear what happens to $\tilde{k}_G((0, z_2), (0, w_2))$.\hfill[7]

Nevertheless, there is the following result (cf. [Par 2003]).

Proposition 2.3.2. For any $n \geq 3$ there exists a pseudoconvex balanced domain $G \subset \mathbb{C}^n$ such that

- $G$ is Brody–hyperbolic,
- $G$ is not $k_G$–hyperbolic.

Proof. Obviously, it suffices to construct such an example $G$ in $\mathbb{C}^3$. Then, in the general case, $G \times E^{n-3}$ will do the job in $\mathbb{C}^n$.

So let $n = 3$. Put $r_j := e^j$, $s_j := 1/(r_j^2 + r_j)$, $t_j := \sqrt{j}/s_j$, $\varepsilon_j := 2^{-j-1}$, and $\eta_j := t_j s_j$, $j \in \mathbb{N}$. Then

$$\sum_{j=1}^{\infty} \varepsilon_j = 1/2, \quad \sum_{j=1}^{\infty} \varepsilon_j \log \frac{1}{\eta_j} \geq \sum_{j=1}^{\infty} \varepsilon_j \log \frac{1}{t_j} > -\infty.$$

For $j \in \mathbb{N}$ define

$$Q_j(z) := z_1 z_2 - s_j(z_3 - z_2)(z_3 - 2z_2), \quad z = (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Put

$$G := \{z \in \mathbb{C}^3 : h(z) < 1\} \text{ with } h(z) := \max\{|z_1|, |z_2|/2, h_0(z)|}.$$
where
\[ h_0(z) := \prod_{j=1}^{\infty} \left| \frac{Q_j(z)}{\eta_j} \right|^{\epsilon_j} = \exp \left( \sum_{j=1}^{\infty} \epsilon_j \log \left| \frac{Q_j(z)}{\eta_j} \right| \right). \]

We claim that \( G \) is a pseudoconvex balanced domain that is Brody–hyperbolic, but not \( \tilde{k} \)-hyperbolic.

**Step 1.** \( h \) is absolutely homogeneous and positive definite.

It suffices to discuss \( h_0 \). Fix \( z \in \mathbb{C}^3 \) and \( \lambda \in \mathbb{C} \). Then:
\[
\sum_{j=1}^{\infty} \epsilon_j \log \left| \frac{Q_j(\lambda z)}{\eta_j} \right| = \sum_{j=1}^{\infty} \epsilon_j \log(|\lambda|^2) + \sum_{j=1}^{\infty} \epsilon_j \log \left| \frac{Q_j(z)}{\eta_j} \right| = \log |\lambda| + \sum_{j=1}^{\infty} \epsilon_j \log \left| \frac{Q_j(z)}{\eta_j} \right|.
\]

Hence, \( h_0(\lambda z) = |\lambda| h_0(z) \).

Assume now that \( h(z) = 0 \). Then \( z_1 = z_2 = 0 = h_0(z) \) which implies that
\[
-\infty = \sum_{j=1}^{\infty} \epsilon_j \log \left| \frac{Q_j(0,0,z_3)}{\eta_j} \right| = \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log(|z_3|^2),
\]
from which we obtain that \( z_3 = 0 \). Hence, \( h \) is positively defined.

**Step 2.** \( h_0 \in \mathcal{PSH}(\mathbb{C}^3) \) (in particular, \( G \) is pseudoconvex).

Fix a positive \( R \) and let \( z \in (RE)^3 \). Then \( |Q_j(z)| \leq (1+6)R^2 \). Recall that \( \eta_j \to \infty \).

Therefore, there is a \( j_R \) such that
\[ |Q_j(z)|/\eta_j < 1, \quad z \in (RE)^3, \quad j \geq j_R. \]

So it follows that \( h_0 \in \mathcal{PSH}((RE)^3) \) for arbitrary \( R \). Hence, \( h_0 \in \mathcal{PSH}(\mathbb{C}^3) \).

**Step 3.** \( G \) is not \( \tilde{k} \)-hyperbolic.

Let
\[ \varphi_j \in O(\mathbb{C}, \mathbb{C}^3), \quad \varphi_j(\lambda) := (s_j\lambda(\lambda - 1), 1, \lambda + 1), \quad j \in \mathbb{N}. \]

Observe that \( Q_j \circ \varphi_j = 0 \) on \( \mathbb{C}, \quad j \in \mathbb{N} \). Therefore, \( \varphi_j(\lambda) \in G \) if \( |\lambda| < r_j \). In particular,
\[
\tilde{k}_G((0,1,1),(0,1,2)) = \tilde{k}_G(\varphi_j(0),\varphi_j(1)) \leq k_E(0,1/r_j) \to 0, \quad j \to \infty,
\]
meaning that \( G \) is not \( \tilde{k} \)-hyperbolic.

**Step 4.** \( G \) is Brody–hyperbolic.

Let \( f = (f_1, f_2, f_3) \in O(\mathbb{C}, G) \). In virtue of the form of \( G, \) \( f_j \) is bounded and so \( f_j \equiv a_j, \quad j = 1, 2. \) Suppose that \( f_3 \) is not constant. Then, in virtue of Picard’s theorem, we have \( \mathbb{C} \setminus \{w\} \subset f_3(\mathbb{C}) \) for a suitable \( w \in \mathbb{C} \). Hence, \( h(a_1, a_2, \cdot) < 1 \) on \( \mathbb{C} \setminus \{w\} \).

Using Liouville’s theorem for subharmonic functions, we conclude that \( h_0(a_1, a_2, \cdot) \equiv \) const. Note that \( h_0(a_1, a_2, \lambda) = 0 \) if \( Q_j(a_1, a_2, \lambda) = 0 \) for at least one \( j \). Therefore, \( h_0(a_1, a_2, \cdot) \equiv 0 \).

To get a contradiction we discuss different cases of \( a_1, a_2 \).

**Case** \( a_2 = 0 \): Then \( Q_j(a_1, 0, \lambda) = -s_j\lambda^2, \quad j \in \mathbb{N}. \) Therefore,
\[
\log h_0(a_1, 0, 1) = \sum_{j=1}^{\infty} \epsilon_j \log \left| \frac{Q_j(a_1,0,1)}{\eta_j} \right| = \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} > -\infty;
\]
contradiction.
Case $a_2 \neq 0$, $a_1 = 0$: Then $Q_j(0, a_2, 0) = -2s_ja_2^2$, $j \in \mathbb{N}$. Therefore,
\[
\log h_0(0, a_2, 0) = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{1}{t_j} + \log(2|a_2|^2) \sum_{j=1}^{\infty} \varepsilon_j > -\infty;
\]
contradiction.

Case $a_1a_2 \neq 0$: Then
\[
\log h_0(a_1, a_2, a_2) = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|a_1a_2|}{\eta_j} > -\infty;
\]
contradiction.

Hence, $G$ is Brody–hyperbolic. \hfill \Box

Remark 2.3.3. It remains an open question whether such an example does exist in $\mathbb{C}^2$.

2.4. Hyperbolcities for Hartogs type domains

Let $G \subset \mathbb{C}^n$ be an arbitrary domain. A domain $D = D(G) \subset G \times \mathbb{C}^n$ is called a Hartogs domain over $G$ with $m$–dimensional balanced fibers if for any $z \in G$ the fiber $D_z := \{w \in \mathbb{C}^n : (z, w) \in D\}$ is a non empty balanced domain in $\mathbb{C}^n$. Recall that for such a $D$ there exists exactly one upper semicontinuous function $H : G \times \mathbb{C}^n \rightarrow [0, \infty)$, $H(z, \lambda w) = |\lambda|H(z, w)$, $z \in G$, $w \in \mathbb{C}^m$, $\lambda \in \mathbb{C}$, such that
\[
D_H = D = \{(z, w) \in G \times \mathbb{C}^m : H(z, w) < 1\}.
\]

Conversely, any such $H$ leads to a Hartogs domain over $G$ with $m$–dimensional balanced fibers.

Recall that $D = D_H$ is pseudoconvex iff $G$ is pseudoconvex and $\log H \in \overline{PSH}(G \times \mathbb{C}^m)$.

Then we have the following hyperbolicity criterion (cf. [DDT-Tho 1998], see also [DDT-PVD 2000]).

Theorem 2.4.1. Let $D = D_H \subset G \times \mathbb{C}^m$ be a Hartogs domain over $G \subset \mathbb{C}^n$ with $m$–dimensional balanced fibers. If $D$ is $k$–hyperbolic, then $G$ is $k$–hyperbolic and, for any compact set $K \subset G$, the function $\log H$ is bounded from below on $K \times \partial \mathbb{B}_m$.

Proof. If $D$ is $k$–hyperbolic then $k_G(z', z'') \geq k_D((z', 0), (z'', 0)) > 0$ for all $z', z'' \in G$, $z' \neq z''$. Hence $G$ is $k$–hyperbolic.

Assume now that there are two sequences $(z^j)_{j \in \mathbb{N}} \subset G$, $\lim z^j =: z^0 \in G$, $(w^j)_{j \in \mathbb{N}} \subset \partial \mathbb{B}_m$, $\lim w^j = w^0 \in \partial \mathbb{B}_m$ such that $\lim_{j \rightarrow \infty} H(z^j, w^j) = 0$. We may assume that $(z^j, w^j) \in D$, $j \in \mathbb{N}$. Then $\varphi_j \in \mathcal{O}(\mathbb{C}, G \times \mathbb{C}^m)$, $\varphi_j(\lambda) := (z^j, \lambda w^j)$, maps $R_jE$ into $D$ for a suitable sequence $(R_j)_{j \in \mathbb{N}}$, with $R_j \rightarrow \infty$. Therefore,
\[
k_D((z^0, 0), (z^j, w^j)) = k_D(\varphi_j(0), \varphi_j(1)) \leq k_E(0, 1/R_j) \rightarrow 0;
\]
hence, $k_D((z^0, 0), (z^0, w^0)) = 0$; contradiction. \hfill \Box
Remark 2.4.2. It seems to be not known whether the converse of Theorem 2.4.1 also holds. Nevertheless, the following special case is true (cf. [DDT-Tho 1998]).

Proposition 2.4.3. Let \( G := E \) and \( u : E \rightarrow (-\infty, \infty) \) upper semicontinuous. Put \( H(z, w) := |w|^{u(z)} \) and \( D := D_H \). Assume that \( u \) is locally bounded from below. Then \( D \) is \( k \)-hyperbolic.

Proof. In virtue of Theorem 7.2.2 in [J-P 1993], it suffices to show that the Kobayashi–Royden pseudometric is locally positive definite, i.e. for any point \( p_0 = (z_0, w_0) \in D \) there exist a neighborhood \( U = U(p_0) \subset D \) and a positive number \( C \) such that \( \kappa_D(p; X) \geq C\|X\|, p \in U, X \in \mathbb{C}^2 \).

First, observe that
\[
g(r) := -\inf \{u(\lambda) : \lambda \in E, |\lambda| \leq r, \quad r \in (0, 1)\}.
\]

Now, let \( s \in (0, 1) \) and fix \( (z_0, w_0) \in D \), \( |z_0| < s \), and \( X \in \mathbb{C}^2 \setminus \{0\} \). Let \( f \in \mathcal{O}(E, D) \) with \( f(0) = (z_0, w_0) \) and \( af'(0) = X \) for \( \alpha \in \mathbb{C}_+ \). In virtue of the Schwarz Lemma, we see that \( |f'(0)| \leq 1 - |z_0|^2 \leq 1 \).

Put \( r_0 := \frac{1 + |z_0|^2}{2} \). Applying the Schwarz Lemma, it follows that, if \( |f_1(\lambda)| \geq r_0 \), then
\[
|\lambda| \geq \frac{|z_0 - f_1(\lambda)|}{1 - \frac{1}{|z_0|f_1(\lambda)}} \geq \frac{|f_1(\lambda)| - |z_0|}{1 - |f_1(\lambda)||z_0|} \geq \frac{r_0 - |z_0|}{1 - r_0|z_0|} \geq \frac{1}{2}.
\]

Put \( \Omega := \{\lambda \in E : |f_1(\lambda)| < r_0\} \). Then \( \sup_{\Omega} |f_2| \leq e^{g(r_0)} \) and \( \mathbb{B}(0, 1/2) \subset \Omega \). Thus, \( |f_2'(0)| \leq 2e^{g(r_0)} \). Then
\[
|\alpha| \geq \max \left\{|X_1|, \left|\frac{X_2}{2e^{g(r_0)}}\right| \right\} \geq \frac{1}{\sqrt{2}} \min \left\{1, \frac{1}{2e^{g(r_0)}} \right\}\|X\|.
\]

Since \( f \) was arbitrarily chosen we get
\[
\kappa_D((z, w); X) \geq \frac{1}{\sqrt{2}} \min \left\{1, \frac{1}{2e^{g(r_0)}} \right\}\|X\|, \quad (z, w) \in D, \ |z| < s.
\]

Hence, \( D \) is \( k \)-hyperbolic.

Remark 2.4.4. In Remark 2.2.5 we mentioned that, if a Reinhardt domain in \( \mathbb{C}^2 \) is \( e \)-hyperbolic, then its envelope of holomorphy is also \( e \)-hyperbolic. In the class of Hartogs domains and the case of \( k \)-hyperbolicity, such a conclusion is false even in dimension 2 (cf. [Die-Hai 2003]).

Let \( u : [0, 1) \rightarrow (-\infty, 0) \) be continuous function satisfying \( \lim_{t \rightarrow 1} \varphi(t) = -\infty \). Put \( u(z_1) := \varphi(\{|z_1|\}) \). Then the domain
\[
D := \{z \in E \times \mathbb{C} : |z_2| < e^{-u(z_1)}\}
\]
is \( k \)-hyperbolic (see Proposition 2.4.3). Recall that
\[
\mathcal{H}(D) = \{z \in E \times \mathbb{C} : |z_2| < \hat{u}(z_1)\},
\]
where \( \hat{u} \) is the largest subharmonic minorant of \( u \). In virtue of the maximum principle for subharmonic function, it is clear that \( \hat{u} \equiv -\infty \). Therefore, \( \mathcal{H}(D) = E \times \mathbb{C} \) which is not \( k \)-hyperbolic.
Remark 2.4.5. So far, we discussed hyperbolicity. We close this part by a remark on the opposite situation. There is the following result due to E. Fornæss and N. Sibony [For-Sib 1981]: Let $D \subset \mathbb{C}^2$ be a domain which can be monotonically exhausted by domains $D_j$, where each of the $D_j$ is biholomorphically equivalent to $\mathbb{B}_2$. If $\partial D \neq 0$, then $D$ is biholomorphically equivalent either to $\mathbb{B}_2$ or to $E \times \mathbb{C}$. Observe that $\mathbb{B}_2(0,j) \not\sim \mathbb{C}^2$, $\mathbb{C}^2 \equiv 0$, but, obviously, $\mathbb{C}^2$ is neither biholomorphic to $\mathbb{B}_2$ nor to $E \times \mathbb{C}$.

It turned out that there is a domain $D \subset \mathbb{C}^n$, $n \geq 2$, $D_j \not\sim D$, each $D_j$ is biholomorphically equivalent to $\mathbb{B}_n$, such that

- $\partial D \equiv 0$,
- $\exists u \in \mathcal{PSH}(\mathbb{C}^n) : D = \{ z \in \mathbb{C}^n : u(z) < 0 \}$ and $u|_D \not\equiv \text{const}$; in particular, $D$ is not biholomorphic to $\mathbb{C}^n$.

Domains of that type are called short $\mathbb{C}^n$’s (see [For 2001]). The domain $D$ is obtained in the following way: Let $d \in \mathbb{N}$, $d \geq 2$, and $\eta > 0$. Denote by $\text{Aut}_{d,\eta}$ the set of all polynomial automorphisms $\Phi$ of $\mathbb{C}^n$ of the form

$$\Phi(z) = \Phi(z_1, \ldots, z_n) = (z_1^d + P_1(z), P_2(z), \ldots, P_n(z)),$$

where $\text{deg} P_j \leq d - 1$, $j = 1, \ldots, n - 1$, and where each coefficient of the polynomials $P_j$ has a modulus at most $\eta$. Choosing sufficiently good sequences $a_j \searrow 0$, $a_j \in (0,1)$, and $F_j \in \text{Aut}_{d,\eta}$, where $\eta_j := a_j^d$, $j \in \mathbb{N}$, gives $D$ as

$$D = \{ z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F_k \circ \cdots \circ F_1(z) = 0 \}.$$

2.5. $c$–completeness for Reinhardt domains

In this chapter Carathéodory completeness for Reinhardt domains will be discussed. Recall that a domain $G \subset \mathbb{C}^n$ is called to be $cG$–complete (shortly, $c$–complete) (respectively, $cG$–finitely compact (shortly, $c$–finitely compact)) if $cG$ is a distance and if any $cG$–Cauchy sequence does converge to a point in $G$ (in the standard topology) (respectively, if $cG$ is a distance and if any $cG$–ball with a finite radius is a relatively compact subset of $G$). Moreover, recall that any $cG$–complete domain $G$ is pseudoconvex.

Theorem 2.5.1. Let $G \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then the following conditions are equivalent:

(i) $G$ is $cG$–finitely compact;
(ii) $G$ is $cG$–complete;
(iii) there is no sequence $(z_v)_{v \in \mathbb{N}} \subset G$ with $\sum_{v=1}^{\infty} g_G(z_v, z_{v+1}) < \infty$;
(iv) $G$ is bounded and fulfills the following so called Fu–condition:

$$\text{if } \overline{G} \cap V_j \neq \emptyset, \text{ then } G \cap V_j \neq \emptyset,$$

where $V_j := \{ z \in \mathbb{C}^n : z_j = 0 \}$.

This result is due W. Zwonek ([Zwo 2000a], see also [Zwo 2000b]); earlier partial results can be found in [Pfl 1984] (see also [J-P 1993]) and [Fu 1994].

For the proof of Theorem 2.5.1 we shall need the following three lemmas.

Lemma 2.5.2 ([Zwo 2000a]). Let $G \subset \mathbb{C}^n_+$ be a pseudoconvex Reinhardt domain. Then $k_G = \tilde{k}_G$. In particular, the Lempert function $k_G$ is continuous on $G \times G$. 
2.5. $c$-completeness for Reinhardt domains

Proof. Observe that $T := \log G$ is a convex domain in $\mathbb{R}^n$ and that the mapping

$$T + i\mathbb{R}^n \ni z \overset{\Phi}{\rightarrow} (x_1, \ldots, x_n) \in G$$

is a holomorphic covering. Therefore, for $z, w \in G$ we have

$$\tilde{k}_G(z, w) = \inf\{k_{T+i\mathbb{R}^n}(\tilde{z}, \tilde{w}) : \tilde{z}, \tilde{w} \in T + i\mathbb{R}^n \text{ with } \Phi(\tilde{z}) = z, \Phi(\tilde{w}) = w\} = \inf\{k_{T+i\mathbb{R}^n}(\tilde{z}, \tilde{w}) : \tilde{z}, \tilde{w} \in T + i\mathbb{R}^n \text{ with } \Phi(\tilde{z}) = z, \Phi(\tilde{w}) = w\} = k_G(z, w).$$

Here we have used the theorem of Lempert. □

Lemma 2.5.3. Let $\Omega \subset \mathbb{R}^n$ be an unbounded convex domain which is contained in $\mathbb{R}^n_{(\infty, R)}$ for a certain number $R$. Then, for any point $a \in \Omega$ there exist a vector $v \in \mathbb{R}^n \setminus \{0\}$ and a neighborhood $V = V(a) \subset \Omega$ such that $V + \mathbb{R}v \subset \Omega$.

Proof. Take $v$-1.0. Then the continuity of the Minkowski function $h$ of $\Omega$ and the assumptions on $\Omega$ lead to a vector $v$ on the unit sphere with $h(v) = 0$. Obviously, $v \in \mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}v \subset \Omega$. Finally, using the convexity of $\Omega$, we see that for any open ball $V \subset \Omega$ with center $a$ the following inclusion holds: $V + \mathbb{R}v \subset \Omega$. □

Lemma 2.5.4 ([Hay-Ken 1976]). Let $H := \{\lambda \in \mathbb{C} : \text{Re } \lambda < 0\}$, $b < 0$, and $M < 0$. Moreover, let $u \in \mathcal{SH}(H)$, $u < 0$, and $u(\lambda) \leq M$ for all $\lambda$ with $\text{Re } \lambda = b$. Then $u \leq M$ on $\{\lambda \in \mathbb{C} : \text{Re } \lambda \leq b\}$.

Now we are in the position to proceed with the proof of the above theorem.

Proof of Theorem 2.5.1. Observe that the following two implications (i) $\implies$ (ii) $\implies$ (iii) are obvious. Moreover, the proof of (iv) $\implies$ (i) follows along the same lines as the one of Theorem 7.4.6 in [J-P 1993].

Therefore, we shall prove only (iii) $\implies$ (iv): Suppose that this implication is false. Then, in virtue of Proposition 2.2.7, we may assume that $G$ is bounded and doesn’t fulfill the Fu-condition (2.5.4). Moreover, without loss of generality we only have to deal with the following situation:

$$\overline{G} \cap V_j \neq \emptyset, \text{ but } G \cap V_j = \emptyset, \ j = 1, \ldots, k,$$

$$\overline{G} \cap V_j = \emptyset, \ j = k + 1, \ldots, n, \ 1 \leq k \leq n.$$ 

In fact, if $G \cap V_j \neq \emptyset$, then one can go to the intersection of $G$ with those coordinate axes.

Hence $G \subset \mathbb{C}^n_+$. We may also assume that $(1, \ldots, 1) \in G$.

Observe that $\log G$ is convex, bounded into all positive directions, unbounded into the first $k$ negative directions, and bounded in the remaining negative directions. Thus, in virtue of Lemma 2.5.3 we find a small ball $V = V(0) \subset \log G$ with center 0 and a vector $v \in \mathbb{R}^n \setminus \{0\}$ such that $V + \mathbb{R}v \subset \log G$. It is clear that $v_j = 0, j = k + 1, \ldots, n$. Without loss of generality, we may assume that $v_j < 0, j = 1, \ldots, \ell$, where $\ell \leq k, v_1 = -1$, and $v_{\ell+1} = \ldots v_n = 0$. Hence,

$$(e^{x_1e^{-t}}, e^{x_2e^{-t}}, \ldots, e^{x_n e^{tv_n}}) \in G, \quad t > 0, \ x \in V.$$
Then, with $\alpha := -v$, we have an $\varepsilon > 0$ such that
\[(e^{\lambda}, \mu_2 e^{\lambda_2}, \ldots, \mu_\ell e^{\lambda_\ell}, 1, \ldots, 1) \in G, \quad \lambda \in H, \ e^{-\varepsilon} < |\mu_j| < e^{\varepsilon}, \ j = 2, \ldots, \ell.\]
Put
\[A := \{(\mu_2, \ldots, \mu_\ell) \in \mathbb{C}^{\ell-1} : e^{-\varepsilon} < |\mu_j| < e^{\varepsilon}, \ j = 2, \ldots, \ell\},\]
\[H_\mathbb{R} := \{\lambda \in \mathbb{C} : \text{Re} \lambda < R\}, \ R \geq 0,\]
\[\Phi : \mathbb{C} \times A \rightarrow \mathbb{C}^\ell, \quad \Phi(\lambda, \mu) := (e^{\lambda}, \mu_2 e^{\lambda_2}, \ldots, \mu_\ell e^{\lambda_\ell}).\]

It is clear that $\Phi(H_\mathbb{R} \times A) =: D_R \subset \mathbb{C}_x^\ell, \ R \geq 0$, is a pseudoconvex Reinhardt domain with $D_R \xrightarrow{R \rightarrow \infty} D_\infty := \Phi(\mathbb{C} \times A) \subset \mathbb{C}_x^\ell$. Therefore,
\[\tilde{k}_{D_{\infty}}(\Phi(-1, 1, \ldots, 1), (\Phi(1, \ldots, 1), z)) = 0 \text{ for all } z \in D_\infty \cap M, \text{ where } M := \Phi(\mathbb{C} \times \{(1, \ldots, 1)\}).\]

Observe that $D_0 \times \{(1, \ldots, 1)\} \subset G$, but $(0, \ldots, 0, 1, \ldots, 1) \notin G$. Now choose positive numbers $a_j, \ j \in \mathbb{N}$, with $\sum_{j=1}^{\infty} a_j < \infty$. It suffices to find points $z^j \in D_0, \ j \in \mathbb{N}$, with $\lim_{j \rightarrow \infty} z^j = 0$ such that
\[g_D((z^j, 1, \ldots, 1), (z^{j+1}, 1, \ldots, 1)) \leq g_{D_0}(z^j, z^{j+1}) \leq a_j, \ j \in \mathbb{N}.
Applying that $\tilde{k}_{D_R}$ is continuous on $G_R \times G_R$, the theorem of Dini, and that
\[
\lim_{R \rightarrow \infty} \tilde{k}_{D_R}(\Phi(-1, 1, \ldots, 1), z) = \tilde{k}_{D_\infty}(\Phi(-1, 1, \ldots, 1), z) = 0,
\]
\[z \in D_\infty \cap \Phi(\mathbb{C} \times \{(1, \ldots, 1)\}), \ e^{-2} < |z_1| < e^{-1},\]
we conclude that this convergence is a uniform one.
Hence we have a sequence $(R_j)_{j \in \mathbb{N}}, \lim_{j \rightarrow \infty} R_j = \infty$, such that
\[\tilde{k}_{D_{R_j}}(\Phi(-1, 1, \ldots, 1), (\Phi(1, \ldots, 1), z^j) < a_j, \ -2 \leq \text{Re} \lambda \leq -1.
Observe that the mapping $\psi_R : D_0 \rightarrow D_R, \ \psi(z) := (e^{R z_1}, z_2 e^{\alpha_2 R}, \ldots, z_\ell e^{\alpha_\ell R}),$ is biholomorphic. Therefore,
\[\tilde{k}_{D_{R_j}}(\Phi(-1 - R_j, 1, \ldots, 1), (\Phi(1, \ldots, 1), z^j) < a_j, \ -2 - R_j \leq \text{Re} \lambda \leq -1 - R_j.
Define
\[u_j(\lambda) := \log g_{D_0}(\Phi(-1 - R_j, 1, \ldots, 1), (\Phi(1, \ldots, 1), \lambda \in H_0).
Observe that $u \in S\mathcal{H}(H_0)$. In virtue of Lemma 2.5.4 it follows that $u(\lambda) < \log a_j$ whenever $\text{Re} \lambda \leq -1 - R_j$. Therefore, we may take $z^j := \Phi(-1 - R_j, 1, \ldots, 1)$ as the desired point-sequence.

\[\square\]

Remark 2.5.5. Obviously, any $c_G$-finitely compact domain $G$ is $c_G$-complete. We point out that the converse (due to N. Sibony and M.A. Selby) is also known for domains in the plane (see [J-P 1993], Theorem 7.4.7). Whether the two notions for the $c$-completeness do coincide for all bounded domains is still unknown. We only mention
that there is a one-dimensional complex space $X$ that is $c_X$–complete but not $c_X$–finitely compact (see [Jar-Pfl-Vig 1993]).

**Remark 2.5.6.** Let $G \subset \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain, $a \in G$, and $z^0 \in \partial G \cap \mathbb{C}^2$. Then $c_G(a, z) \to \infty$ (see [Zwo 2000b]). So that part of $\partial G$ not lying on an coordinate axis is $c_G$–infinitely far away from any point of $G$.

We point out that this phenomenon remains not true in higher dimensions.

**Example 2.5.7** (cf. [Zwo 2000b]). Let $\alpha > 0$ be an irrational number. Put $$G := \{z \in \mathbb{C}^1 : |z_1||z_2|^{\alpha}|z_3|^{\alpha+1} < 1, |z_2||z_3| < 1, |z_3| < 1\}.$$ Then $G$ is a pseudoconvex Reinhardt domain. Fixing points $z^0 \in G \cap \mathbb{C}^3$ and $w \in \mathbb{C}^3$ with $|w_1||w_2|^{\alpha}|z_3|^{\alpha+1} = 1$, $|w_2|^{-1}|w_3|^{2} < 1$, and $|w_3| < 1$, we get $$\limsup_{G\ni z \to w} c_G(z^0, z) < \infty.$$ Moreover, the biholomorphic map $$\Phi : G \cap \mathbb{C}^3 \to \mathbb{C}^3, \quad \Phi(z) := (z_1 z_2^{[\alpha]+1} z_3^{[\alpha]+3}, z_2^{2} z_3^{2}),$$ has as its image a bounded pseudoconvex Reinhardt $G^*$ domain contained in $$\{z \in \mathbb{C}^3 : |z_2| < 1, |z_3| < 1, |z_1||z_2|^{\alpha-|\alpha|-1}|z_3|^{\alpha-\alpha} < 1\}.$$ In the class of Reinhardt domains we have the following characterization of hyperconvexity (cf. [Zwo 2000a] and [Car-Ceg-Wik 1999]).

**Theorem 2.5.8.** Let $G \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then the following conditions are equivalent:

(i) $G$ is hyperconvex;

(ii) $G$ is bounded and fulfills the Fu–condition.

**Proof.** The direction (ii) $\implies$ (i) follows directly from Theorem 2.5.1(i).

To prove the converse, suppose that $G$ doesn’t fulfill the conditions in (ii). According to Proposition 2.2.7, we may assume that $G$ is bounded and doesn’t fulfill the Fu condition. Hence, without loss of generality, we may assume that $G = D_0$ (compare the proof of Theorem 2.5.1), i.e. $$G := \{\zeta \in \mathbb{C}^n : \zeta \in E_*, \mu_j \in \mathbb{C}, e^{-\varepsilon_0} < |\mu_j| < e^{\varepsilon_0}, j = 2, \ldots, n\},$$ where $\varepsilon_0 > 0$, $\mu_0 > 0$, $\mu_j > 0$, $j = 2, \ldots, n$.

Let $u \in \mathcal{PSH}(G) \cap \mathcal{L}(G)$, $u < 0$, be such that $\{z \in G : u(z) < -\varepsilon\} \subset G$ for any $\varepsilon > 0$. Define $$v(z) := \sup \{u(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) : \theta_j \in \mathbb{R}\}.$$ Obviously, $v$ is an exhausting function of $G$ with $v(z) = v(|z_1|, \ldots, |z_n|)$. Therefore, the function $$E_* \ni \lambda \rightarrow v(|\lambda|, |\lambda|^2, \ldots, |\lambda|^\alpha)$$ is subharmonic and bounded from above by 0. Hence it can be continued as a function $v^* \in \mathcal{SH}(E)$. Then, in virtue of the hyperconvexity of $G$, it follows that $v^*(0) = 0$ implying that $v = 0$ — contradiction. \qed
Recall that the Carathéodory distance is not inner. So, in general, we have $c_G \leq c^i_G$, $c_G \neq c^i_G$ (cf. §1.2.1). Moreover, it is known that $c^i_G = \int \gamma_G = \int \gamma_G^{(1)} \leq \int \gamma_G^{(k)}, k \in \mathbb{N}$. Thus,

$$c_G - \text{complete } \implies c^i_G - \text{complete } \implies \int \gamma_G^{(k)} - \text{complete}, \quad k \in \mathbb{N}.$$  

(A domain $G$ is called $\delta_G - \text{complete}$ (shortly, $\delta - \text{complete}$) if $\delta_G$ is a distance and any $\delta_G$-Cauchy sequence in $G$ does converge to a point in $G$, $\delta \in \{c^i, \int \gamma^{(k)}, k \in \mathbb{N}\}. \quad$ See [J-P 1993] for more details.) In fact there is the following result (see [Zwo 2001b], [Zap 2003]).

**Theorem 2.5.9.** Let $G \subset \mathbb{C}^n$ be a bounded pseudoconvex Reinhardt domain. Then the following properties are equivalent:

(i) $G$ is $\mathcal{C}_G$-complete;

(ii) $G$ is $\mathcal{C}_G^i$-complete;

(iii) $G$ is $\int \gamma_G^{(k)}$-complete, $k \in \mathbb{N}$;

(iv) there is a $k \in \mathbb{N}$ such that $G$ is $\int \gamma_G^{(k)}$-complete.

In order to be able to prove Theorem 2.5.9, we first recall a fact on multi-dimensional Vandermonde’s determinants (for example, see [Sic 1962]), namely:

Let $X_s := (s, \ldots, s) \in \mathbb{C}^n$, $s \in \mathbb{N}$, and $N_k := \#\{\alpha \in \mathbb{Z}_+^n : |\alpha| \leq k\}, k \in \mathbb{N}$. Then

$$\det (X_s^{1 \leq s \leq N_k, |\alpha| \leq k}) \neq 0. \quad (2.5.5)$$

Using this information we get the following

**Lemma 2.5.10.** Let $P(z) = \sum_{1 \leq |\beta| \leq k} b_\beta z^\beta, z \in \mathbb{C}^n$, be a polynomial in $\mathbb{C}^n$. Then there are numbers $(N_j)_{1 \leq j \leq k} \subset \mathbb{N}$, $(c_{j,s})_{1 \leq j \leq k, 1 \leq s \leq N_j} \subset \mathbb{C}$, and vectors $(X_{j,s})_{1 \leq j \leq k, 1 \leq s \leq N_j} \subset \mathbb{C}^n$ such that

$$P(z) = \sum_{j=1}^k \sum_{s=1}^{N_j} c_{j,s} \sum_{|\beta|=j} \frac{n!}{\beta!} p_\beta(z) X^\beta_{j,s}, \quad z \in \mathbb{C}^n,$$

where

$$p_\beta(z) := \prod_{j=1}^n p_{\beta,j}(z), \quad p_{\beta,j}(z) := z_j(z_j - 1) \cdots (z_j - \beta_j + 1), \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

**Proof.** The proof is by induction on $k \in \mathbb{N}$. Obviously, the case $k = 1$ is true. So we may assume that Lemma 2.5.10 holds for a $k \in \mathbb{N}$. Now take a polynomial $P(z) = \sum_{1 \leq |\beta| \leq k+1} b_\beta z^\beta, z \in \mathbb{C}^n$, and write

$$P(z) = \sum_{1 \leq |\beta| \leq k} b_\beta z^\beta + \sum_{|\beta|=k+1} b_\beta(z^\beta - p_\beta(z)) + \sum_{|\beta|=k+1} b_\beta p_\beta(z), \quad z \in \mathbb{C}^n.$$

Observe that the first two terms are of degree less than or equal to $k$. The third one may be written as

$$\sum_{|\beta|=k+1} b_\beta p_\beta(z) = \sum_{|\beta|=k+1} \frac{(k+1)!}{\beta!} p_\beta(z) \frac{\beta! b_\beta}{(k+1)!}.$$  

(3) Notice that here $X_s$ is a vector and not the $s$-th coordinate of a vector.
Using (2.5.5), we find $(c_s)_{1 \leq s \leq N_k+1} \subset \mathbb{C}$ and $(X_s)_{1 \leq s \leq N_k+1} \subset \mathbb{C}^n$ such that

$$\frac{\beta! b_\beta}{(k+1)!} = \sum_{s=1}^{N_k+1} c_s X_s^\beta, \quad |\beta| = k + 1.$$ 

Hence, Lemma 2.5.10 has been proved. □

**Proof of Theorem 2.5.9.** It remains to prove $(iv) \implies (i)$. We may assume that $n \geq 2$. Suppose that $G$ doesn’t fulfill the Fu–condition, but $G$ is $\beta$–complete for a suitable $k$. According to the proof of Theorem 2.5.1, we may assume that $G \subset \mathbb{C}^n$ and that

$$\log G = \{0\} \times (\log \delta, -\log \delta) + \mathbb{R}_{>0} v,$$

where $\delta \in (0,1)$, $v \in (-\infty,0)^n$, and $v_1 = -1$. Put $\gamma := -v$.

Observe that a monomial $z^\alpha$ ($\alpha \in \mathbb{Z}^n$) is bounded on $G$, if and only if $\langle \alpha, \gamma \rangle \geq 0$.

Put $\chi : (0,1) \to G$, $\chi(t) := (t^{\gamma_1}, \ldots, t^{\gamma_n})$. For a fixed $t \in (0,1)$, we are going to estimate $\gamma^{(k)}(\chi(t); \chi'(t))$.

Fix an $f \in \mathcal{O}(G,E)$, $\text{ord}_\chi f \geq k$. Then, using Laurent expansion, we get

$$f(z) = \sum a_\alpha z^\alpha,$$

where

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = r_1, \ldots, |\zeta_n| = r_n} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{\zeta^{\alpha+1}}$$

is independent of $r = (r_1, \ldots, r_n) \in G \cap \mathbb{R}^n_{>0}$. Note that

$$|a_\alpha| \leq \frac{1}{r^n} \text{ for any } r \in G. \quad (2.5.6)$$

From (2.5.6) it follows that $a_\alpha = 0$ if $\langle \alpha, \gamma \rangle < 0$. Therefore,

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle \geq 0} a_\alpha z^\alpha, \quad z \in G.$$

Taking $r_1 < 1$ in (2.5.6) arbitrarily large and $r_j$ arbitrarily close to $\delta$ (or to $\delta^{-1}$), $j = 2, \ldots, n$, then

$$|a_\alpha| \leq \delta^{|\alpha_2| + \cdots + |\alpha_n|}.$$

Taking derivatives we have

$$\frac{1}{s!} f^{(s)}(\chi(t))(X_1 t^{\gamma_1 - \frac{k}{s}}, \ldots, X_n t^{\gamma_n - \frac{k}{s}})$$

$$= \sum_{\langle \alpha, \gamma \rangle \geq 0} \left( a_\alpha \sum_{|\beta| = s} \frac{1}{|\beta|!} \beta_\beta(\alpha) X^\beta f^{(\alpha,\gamma) - k} \right), \quad s \in \mathbb{N}, \ X = (X_1, \ldots, X_n) \in \mathbb{C}^n.$$  

Since $\text{ord}_\chi f \geq k$, it follows that

$$\sum_{\langle \alpha, \gamma \rangle \geq 0} a_\alpha f^{(\alpha,\gamma) - k} \sum_{|\beta| = s} \frac{1}{|\beta|!} \beta_\beta(\alpha) X^\beta = 0, \quad 0 \leq s < k, \ X \in \mathbb{C}^n. \quad (2.5.7)$$
Moreover,
\[
\frac{1}{k!}f^{(k)}(\chi(t))(\chi'(t)) = \frac{1}{k!} \sum_{(\alpha,\gamma) \geq 0} a_{\alpha} \langle \alpha, \gamma \rangle^k t^{(\alpha,\gamma)-k} + \frac{1}{k!} \sum_{(\alpha,\gamma) \geq 0} a_{\alpha} \langle \alpha, \gamma \rangle^{-k} \sum_{|\beta|=k} k! \frac{1}{|\beta|!} (p_{\beta}(\alpha) - \alpha^\beta) \gamma^\beta. \tag{2.5.8}
\]

Applying Lemma 2.5.10 and (2.5.7) shows that the second term in (2.5.8) vanishes.
Hence, we have
\[
\frac{1}{k!}f^{(k)}(\chi(t))(\chi'(t)) = \frac{1}{k!} \sum_{(\alpha,\gamma) \geq 0} a_{\alpha} \langle \alpha, \gamma \rangle^k t^{(\alpha,\gamma)-k}.
\]

In virtue of the above estimate, we obtain the following inequality
\[
\gamma^{(k)}_G(\chi(t); \chi'(t)) \leq \frac{1}{\sqrt{k!}} \sum_{\alpha \in \mathbb{Z}^n: (\alpha,\gamma) > 0} \delta^{(|\alpha_2|+\cdots+|\alpha_n|)/k} \langle \alpha, \gamma \rangle t^{((\alpha,\gamma)/k)-1}, \quad t \in (0, 1).
\]

What remains to show is that \( L_{\gamma^{(k)}_G}(\chi|_{[0,1/2]} \) is finite, which would give the desired contradiction.
So we have the following estimate:
\[
L_{\gamma^{(k)}_G}(\chi|_{[0,1/2]} \) \leq \int_0^{1/2} \gamma^{(k)}_G(\chi(t); \chi'(t)) dt \\
\leq \frac{1}{\sqrt{k!}} \sum_{(\alpha,\gamma) \geq 0} \delta^{(|\alpha_2|+\cdots+|\alpha_n|)/k} \int_0^{1/2} \langle \alpha, \gamma \rangle t^{((\alpha,\gamma)/k)-1} dt \\
= \frac{1}{\sqrt{k!}} \sum_{(\alpha,\gamma) > 0} \delta^{(|\alpha_2|+\cdots+|\alpha_n|)/k} \frac{k}{2^{(\alpha,\gamma)/k}} \\
= \frac{k}{\sqrt{k!}} \sum_{\alpha_2,\ldots,\alpha_n \in \mathbb{Z}} \delta^{(|\alpha_2|+\cdots+|\alpha_n|)/k} \sum_{\alpha_1 \in \mathbb{Z}: (\alpha_1,\ldots,\alpha_n) > (\alpha',\ldots,\alpha_n)} \frac{1}{2^{(\alpha',\ldots,\alpha_n)/k}},
\]
where \( \alpha' := (\alpha_2,\ldots,\alpha_n) \), \( \gamma' := (\gamma_2,\ldots,\gamma_n) \). Obviously, the last number is finite, which finishes the proof.

**Remark 2.5.11.** Observe that in the case \( \gamma \in \mathbb{Q}^n \) the above proof may be essentially simplified. Namely, then the punctured unit disc can be embedded into \( G \). So the non-\( \gamma^{(k)}_G \)-completeness of \( G \) follows immediately from the one of \( E_* \).

**Remark 2.5.12.** Let \( G \subset \mathbb{C}^n \) be an arbitrary domain and \( A \subset G \) finite. In generalization of the notion of \( \mathbb{C}^n \)-finitely compact we say that \( G \) is \( m_G(A,\cdot) \)-finitely compact if for any \( R > 0 \) the set \( \{ z \in G : m_G(A, z) < R \} \) is relatively compact in \( G \). Obviously, any
2.7. $f_G^{(k)}$-completeness for Zalcman domains

$m_G(A, \cdot)$–finitely compact domain is $c_G$–finitely compact. Is there a geometrical characterization for $m_G(A, \cdot)$–finite compactness in the class of all pseudoconvex Reinhardt domains as in Theorem 2.5.1?

2.6. $c$–completeness for complete circular domains

Let $G \subset \mathbb{C}^n$ be a bounded pseudoconvex balanced (:= complete circular) domain. Then there is an $h = h_G \in \mathcal{PSH}(\mathbb{C}^n)$ with $h(\lambda z) = |\lambda|h(z)$ ($\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$) such that

$$G = G_h = \{z \in \mathbb{C}^n : h(z) < 1\}$$

and, since $G$ is pseudoconvex, $\log h \in \mathcal{PSH}(\mathbb{C}^n)$ (see Chapter 1).

It is known (due to T. Barth) that $h$ is continuous, whenever $G$ is $k_G$–complete.

In dimensions larger than 2 the converse statement becomes false; in fact there is a counterexample due to Jarnicki–Pflug ([J-P 1993], Theorem 7.5.7). In particular, this example is not $c$–complete.

It is still open which conditions on $h$ may imply that $G = G_h$ is $c$–complete. Moreover, in dimension 2, so far it is not known whether the continuity of $h$ implies the Kobayashi completeness or even the Carathéodory completeness.

2.7. $f_G^{(k)}$–completeness for Zalcman domains

First, we introduce the class of domains we like to study. Let $(a_j)_{j \in \mathbb{N}}$ and $(r_j)_{j \in \mathbb{N}}$ be sequences of positive real numbers such that:

- $2r_j < a_j$, $j \in \mathbb{N}$,
- $a_j \downarrow 0$,
- $\overline{B}(a_j, r_j) \subset E$, $\overline{B}(a_j, r_j) \cap \overline{B}(a_k, r_k) = \emptyset$, $j \neq k$.

Then $G := E \setminus \bigcup_{j=1}^{\infty} \overline{B}(a_j, r_j)$ is called a Zalcman type domain.

The main result here is the following one due to P. Zapalowski (see [Zap 2002] and [Zap 2004]).

**Theorem 2.7.1.** For any $k \in \mathbb{N}$ there exists a Zalcman type domain $G$ which is $f_G^{(l)}$–complete, but not $f_G^{(m)}$–complete, whenever $m \leq k < l$.

**Remark 2.7.2.** It seems to be an open problem whether for different $k, l \in \mathbb{N}$, $k < l$, there exists a Zalcman type domain $G$, which is $f_G^{(k)}$–complete, but not $f_G^{(l)}$–complete.

Note that for $l = sk$, $s \in \mathbb{N}$, it is impossible because of $f_G^{(k)} \leq f_G^{(sk)}$.

Before giving the proof of Theorem 2.7.1 we mention the following sufficient condition for a Zalcman type domain to be not $f_G^{(k)}$–complete.

**Proposition 2.7.3.** Let $G \subset \mathbb{C}$ be a Zalcman type domain (as above) and let $k \in \mathbb{N}$, $\alpha \in (0, 1)$, and $c > 0$. Assume that

$$\gamma_G^{(k)}(t; 1) \leq c|t|^{-\alpha}, \quad t \in (-1, 0).$$

(2.7.9)
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Then $G$ is not $\int \gamma^{(\ell)}_G$–complete for any $\ell \geq k$.

**Proof.** Fix $\ell \in \mathbb{N}$, $\ell \geq k$, and a point $t \in (-1, 0)$. Take an $f \in \mathcal{O}(G,E)$, $f(t) = f'(t) = \ldots = f^{(\ell-1)}(t) = 0$, with $(\gamma^{(\ell)}_G (t; 1))^\ell = \frac{1}{\ell!} |f^{(\ell)}(t)|$. We define

$$g(z) = \begin{cases} \frac{f(z)}{(z-t)^{\frac{k}{\ell}}} & \text{if } z \neq t \\ 0 & \text{if } z = t. \end{cases}$$

Then $g$ is holomorphic with

$$g^{(m)}(t) = 0, \quad m = 0, \ldots, k - 1, \quad \text{and } g^{(k)}(t) = \frac{k!}{\ell!} f^{(\ell)}(t).$$

Moreover, in virtue of the maximum principle, we have

$$\|g\|_G \leq \text{dist}(t, \partial G)^{-(\ell - k)}.$$

Therefore, $h := g \text{dist}(t, \partial G)^{\ell - k} \in \mathcal{O}(G,E)$ and we obtain

$$\left(\gamma^{(k)}_G (t; 1) \right)^{\ell} \geq \frac{1}{\ell!} |h^{(k)}(t)| = \frac{\text{dist}(t, \partial G)^{\ell - k}}{\ell!} |f^{(\ell)}(t)| = \text{dist}(t, \partial G)^{\ell - k} \left(\gamma^{(\ell)}_G (t; 1) \right)^{\ell}. \tag{2.7.10}$$

Finally, from the assumed inequality (2.7.9) the following estimate follows

$$\gamma^{(\ell)}_G (t; 1) \leq \frac{c^{\ell/\ell} |t|^{-\alpha k/\ell}}{|t|^{(\ell - k)/\ell}} = c' |t|^{-\alpha'},$$

where $c' := c^{\ell/\ell}$ and $\alpha' := (\alpha k + (\ell - k))/\ell < 1$. Then integrating along the segment $(-1/2, 0)$ shows that $G$ is not $\int \gamma^{(\ell)}_G$–complete. \qed

Consequently, to find examples as claimed in Theorem 2.7.1 we should try to deal with situations where the boundary behavior of $\gamma^{(k)}_G$ is of the following type

$$\gamma^{(k)}_G (z; 1) \leq c \text{dist}(z, \partial G)^{-1} |\log \text{dist}(z, \partial G)|^{-a}$$

with some $\alpha > 1$, $c > 0$.

**Lemma 2.7.4.** Let $G \subset \mathbb{C}$ be a Zalcman type domain and $k \in \mathbb{N}$. Then there exists a $C > 0$ such that

$$|f^{(k)}(z)| \leq C \left( 1 + \sum_{j=1}^{\infty} \frac{r_j}{(a_j - z)^{k + 1}} \right), \quad z \in (-1/2, 0), \quad f \in \mathcal{O}(G,E).$$

**Proof.** Choose numbers $\tilde{a}_j \in (0, a_j)$ and $\tilde{r}_j \in (\tilde{a}_j, 1)$ such that

$$\mathbb{B}(a_s, r_s) \subset \mathbb{B}(\tilde{a}_j, \tilde{r}_j), \quad s > j, \quad \text{and } \mathbb{B}(\tilde{a}_j, \tilde{r}_j) \cap \mathbb{B}(a_j, r_j) = \emptyset.$$

Put

$$G_j := E \setminus \left( \mathbb{B}(\tilde{a}_j, \tilde{r}_j) \cup \bigcup_{s=1}^{j} \mathbb{B}(a_s, r_s) \right).$$
G_j is a \((j + 2)\)-connected domain with \(G_j \subset G, j \in \mathbb{N}\). Then, for a sufficiently small positive \(\varepsilon_j\) (we may assume that \(\varepsilon_j \xrightarrow{j \to 0} 0\)), we have

\[
G_{j,\varepsilon_j} := (1 - \varepsilon_j)E \setminus \left( B(\bar{a}_j, \bar{r}_j + \varepsilon_j) \cup \bigcup_{s=1}^{j} B(a_s, r_s + \varepsilon_j) \right) \subset G_j.
\]

In virtue of the Cauchy integral, we see that

\[
f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta| = 1 - \varepsilon_j} f(\zeta) \frac{d\zeta}{(\zeta - z)^{k+1}} - \frac{k!}{2\pi i} \int_{|\zeta - \bar{a}_j| = \bar{r}_j + \varepsilon_j} f(\zeta) \frac{d\zeta}{(\zeta - z)^{k+1}}
\]

\[
- \sum_{s=1}^{j} \frac{k!}{2\pi i} \int_{|\zeta - a_s| = r_s + \varepsilon_j} f(\zeta) \frac{d\zeta}{(\zeta - z)^{k+1}}, \quad z \in G_{j,\varepsilon_j}, \quad f \in O(G, E).
\]

Let \(z \in (-1/2, 0)\). Then \(z \in (-1/2, \bar{a}_j - \bar{r}_j - \varepsilon_j - 2^{k+1}\sqrt{\bar{r}_j + \varepsilon_j})\) and \(z < -\varepsilon_j\) for all sufficiently large \(j\). Hence we obtain

\[
|f^{(k)}(z)| \leq \frac{k!}{2\pi} \int_{0}^{2\pi} \frac{1 - \varepsilon_j}{(1 - \varepsilon_j)e^{it} - z} dt + \frac{k!}{2\pi} \int_{0}^{2\pi} \frac{\bar{r}_j + \varepsilon_j}{|\bar{r}_j + \varepsilon_j| e^{it} + \bar{a}_j - z} dt
\]

\[
+ \sum_{s=1}^{j} \frac{k!}{2\pi} \int_{0}^{2\pi} \frac{r_s + \varepsilon_j}{(r_s + \varepsilon_j) e^{it} + a_s - z} dt
\]

\[
\leq k! \left( \frac{1 - \varepsilon_j}{(1/2 - \varepsilon_j)^{k+1}} + \frac{\bar{r}_j + \varepsilon_j}{(\sqrt{\bar{r}_j + \varepsilon_j})^{k+1}} + \sum_{s=1}^{j} \frac{r_s + \varepsilon_j}{(1/2(a_s - z - \varepsilon_j))^{k+1}} \right).
\]

Observe that here the assumption \(2r_j < a_j, j \in \mathbb{N}\), is used to estimate the third term.

Since \(\varepsilon_j \xrightarrow{j \to 0} 0\), we finally receive the following inequality

\[
|f^{(k)}(z)| \leq k! \left( 2^{k+1} + \sqrt{\bar{r}_j} + 2^{k+1} \sum_{s=1}^{j} \frac{r_s}{(a_s - z)^{k+1}} \right).
\]

Recall that \(\sqrt{\bar{r}_j} \xrightarrow{j \to \infty} 0\). Therefore, letting \(j \to \infty\), we obtain

\[
|f^{(k)}(z)| \leq k! 2^{k+1} \left( 1 + \sum_{s=1}^{\infty} \frac{r_s}{(a_s - z)^{k+1}} \right).
\]

**Lemma 2.7.5.** For every \(k \in \mathbb{N}\) there are a \(\tilde{k} \in \mathbb{N}\) and a Zalcman type domain \(G\) such that

(a) \(\limsup_{z \to (1/2)\sqrt{2^{\tilde{k}-1}, z}} f^{(m)}_{G}(z) = \infty, \quad 1 \leq m \leq k\),

(b) \(\lim_{z \to (1/2)\sqrt{2^{\tilde{k}}, z}} f^{(\ell)}_{G}(w, z) = \infty, \quad w \in G, k < \ell\).

Observe that Lemma 2.7.5 implies immediately Theorem 2.7.1.
Proof of Theorem 2.7.1. Fix a $k \in \mathbb{N}$ and take the Zalcman type domain from Lemma 2.7.5. Let $m \in \mathbb{N}$, $1 \leq m \leq k$. As a direct consequence of (a) and the fact that the $f_{\gamma_G^{(m)}}$-completeness is equivalent to the $f_{\gamma_G^{(m)}}$-finite compactness we see that $G$ is not $f_{\gamma_G^{(m)}}$-complete.

It remains to see that $G$ is $f_{\gamma_G^{(\ell)}}$-complete whenever $\ell > k$. So let us fix such an $\ell$, a point $w \in G$, and a boundary point $z^0 \in \partial G$. We have to show that $\lim_{z \to z^0} (f_{\gamma_G^{(\ell)}})(w, z) = \infty$.

Case 1: If $z^0 = 0$, then using (b) we are done.

Case 2: If $|z^0| = 1$, it follows that

$$\lim_{z \to z^0} (f_{\gamma_G^{(\ell)}})(w, z) \geq \lim_{z \to z^0} (f_{\gamma_E^{(\ell)}})(w, z) = \lim_{z \to z^0} c_E(w, z) = \infty.$$

Case 3: If $z^0 \in \partial B(a_j, r_j)$ for some $j$, then

$$\lim_{z \to z^0} (f_{\gamma_G^{(\ell)}})(w, z) \geq \lim_{z \to z^0} c_G(w, z) \geq \lim_{z \to z^0} c_{E \cap B(a_j, r_j)}(w, z) = \lim_{z \to z^0} c_E(\frac{r_j}{w - a_j}, \frac{r_j}{z - a_j}) = \infty,$$

since $|r_j/(z - a_j)| \to 1$ as $z \to z^0$. \qed

What remains is the proof of Lemma 2.7.5.

Proof of Lemma 2.7.5. Let $k \in \mathbb{N}$, $a_j := 2^{-j}$, and $r_{k,j} := 2^{-j} \cdot 2^{k-1}$, $j \in \mathbb{N}$. Since $\lim_{s \to \infty} (\frac{s}{2^j})^2 \cdot \frac{1}{2^j} < 1$, we may choose a $k \in \mathbb{N}$ such that $(\frac{s}{2^j})^2 \cdot \frac{1}{2^j} < 1$, $s \geq \tilde{k}$. Put

$$G_k := E \setminus \bigcup_{j \geq k} B(a_j, r_{k,j}).$$

Obviously, $G_k$ is a Zalcman type domain.

To prove (a) it suffices to verify the following inequality:

$$\exists_{c = c(k) > 0} : \gamma_{G_k^{(m)}}(z; 1) \leq \frac{c}{-z(-\log(-z))^{\frac{m+1}{m+2}}}, \quad z \in [-\frac{1}{2k+1}, 0), \quad m \leq k. \quad (2.7.11)$$

In fact, let $z \in [-\frac{1}{2k+1}, 0)$. Then there exist a unique $N \in \mathbb{N}$, $N \geq \tilde{k}$, and a $b \in (1, 2]$ such that $z = -b/2^N$. Therefore,

$$\sum_{j = k}^{N} \frac{r_{k,j}}{(a_j - z)^{m+1}} \leq \sum_{j = k}^{N} \frac{r_{k,j}}{a_j^{m+1}} = \sum_{j = k}^{N} \frac{2^{jm}}{j^{k+1}} \leq \frac{2^{Nm}}{N^{k+1}} \frac{\delta}{1 - \delta} \frac{2^{Nm}}{N^{k+1}},$$

$$\sum_{j = N}^{\infty} \frac{r_{k,j}}{(a_j - z)^{m+1}} \leq \sum_{j = N}^{\infty} \frac{r_{k,j}}{(-z)^{m+1}} = \sum_{j = N}^{\infty} \frac{2^{N(m+1)}}{2^{j(k+1)m+1}} \leq \frac{2^{N(m+1)}}{N^{k+1}} \frac{1}{2^j} \leq \frac{2^{Nm+1}}{N^{k+1}}. \quad (2.7.12)$$

(The second inequality in (2.7.12) follows easily from the observation that there is a positive $\delta < 1$ such that $\frac{\delta^{N(m+1)}}{s^{m+2}} \leq \delta \frac{2^{Nm}}{s^{m+2}}, \quad s \geq \tilde{k}, \quad m \leq k.$)
Thus the constant $C$ (obviously, $f = \alpha$ where $N$ find an $f$ where $\ell > k$), we claim that $f \in \mathcal{O}(G, E)$. Assume for a while that (2.7.14) is correct. Fix an $\ell > k$ and a $\gamma$-curve $\alpha : [0, 1] \rightarrow G_k$ connecting $z$ with $w$. Then we have

$$\int_0^1 \frac{c}{\Gamma(\alpha(t); \alpha'(t))} dt \geq c \int_0^{t_\alpha} \frac{\alpha'(t) dt}{\alpha(t) \log(1/|\alpha(t)|)} \geq c \int_0^{t_\alpha} \frac{d}{dt} \left( - \log \log(1/|\alpha(t)|) \right) dt = c \left( \log \log \frac{1}{|z|} - \log \log 2^{k-2}, \right),$$

where $t_\alpha := \sup \{ t \in [0, 1] : |\alpha(t)| < \frac{1}{2^{\delta}}, 0 \leq \tau \leq t \}$.

Since the curve $\alpha$ was an arbitrary one connecting $z$ and $w$ in $G_k$, it follows that

$$f^{t(\ell)}(w, z) \geq c \log \log \frac{1}{|z|} - \log \log 2^{k-2} \quad \text{as} \quad z \rightarrow 0,$$

Hence, (b) is verified.

What remains is the proof of (2.7.14). Fix a $z \in G_k \cap \mathbb{B}(0, \frac{1}{2^{\delta}})$. Then we have to find an $f \in \mathcal{O}(G_k, E)$ satisfying the following conditions:

- $f(z) = f'(z) = \ldots = f^{t(\ell-1)}(z) = 0$,
- $|f^{t(\ell)}(z)| \geq \frac{c}{(\log(1/|z|))}$, where $c$ is independent of $z$.

Again we write $z = b e^{i\theta} / 2^N$ with $N \in \mathbb{N}$, $b \in (1, 2]$, and $\theta \in [0, 2\pi)$. Observe that $N \geq \tilde{k} - 1$. Put

$$f(\lambda) := \sum_{j=0}^{t-1} \alpha_{b, \theta, j} (2^{-N-j-1} - \lambda)^{-1} + 2^{N+1} \beta_{b, \theta}, \quad \lambda \in G_k,$$

where $\alpha_{b, \theta, 0} := 1$ and $\alpha_{b, \theta, 1}, \ldots, \alpha_{b, \theta, t-1}, \beta_{b, \theta} \in \mathbb{C}$ depend only on $b$ and $\theta$ such that (obviously, $f \in \mathcal{O}(G_k)$) $f(z) = \ldots = f^{t(\ell-1)}(z) = 0$. 

In virtue of Lemma 2.7.4, we obtain

$$|f^{t(m)}(z)| \leq C \left( 1 + \frac{C_1}{(-z)^m \log(1/|z|)} \right) \leq \frac{2CC_1}{(-z)^m \log(1/|z|)}, \quad f \in \mathcal{O}(G, E),$$

which finally proves (2.7.11). (Note that we may take $C = \hat{k}2^{k+1} \geq m2^{m+1}, m \leq k$; thus the constant $C = C(k)$ from Lemma 2.7.4 works for all $m, m \leq k$.)
Proof. Let α be a function as in (2.7.14) with unknown numbers α_{b,θ,j}, \ j = 1, \ldots, ℓ - 1, and β_{b,θ}, such that

- \max\{α_{b,θ,j} : j = 1, \ldots, ℓ - 1, β_{b,θ}, b and θ as above \} ≤ α,
- \min\{|B_{ℓ,b,θ}| : b ∈ [1, 2], θ ∈ [0, 2π]\} ≥ B_ℓ,
- f(z) = f' (z) = \ldots = f^{(ℓ-1)} (z) = 0 \ (\text{for } f \text{ see (2.7.15)).}

Then the condition f'(z) = \ldots = f^{(ℓ-1)} (z) = 0 gives the following system of ℓ - 1 equations

\[ \sum_{j=0}^{ℓ-1} s! \left( \frac{2^{N+j+1}}{1 - 2^{j+1}b_θ} \right)^{s+1} α_{b,θ,j} = 0, \ s = 1, \ldots, ℓ - 1, \]

which is equivalent to

\[ \sum_{j=1}^{ℓ-1} s! \left( \frac{2^j}{1 - 2^{j+1}b_θ} \right)^{s+1} α_{b,θ,j} = - \left( \frac{1}{1 - 2b_θ^{1/2}} \right)^{s+1}, \ s = 1, \ldots, ℓ - 1. \quad (2.7.16) \]

To simplify further discussions we put

\[ A_{b,θ,j} := \frac{2^j}{1 - 2^{j+1}b_θ}, \quad j = 0, \ldots, ℓ - 1. \]

In order to finish the proof of Lemma 2.7.4 we need the following lemma.

Lemma 2.7.6. For an ℓ ∈ \mathbb{N} there are positive numbers α and B_ℓ such that for every \ z = be^{iθ}/2^N, where b ∈ [1, 2], θ ∈ [0, 2π], and N ≥ ℓ - 1, there exist complex numbers α_{b,θ,j}, j = 1, \ldots, ℓ - 1, and β_{b,θ} such that

- \max\{α_{b,θ,j} : j = 1, \ldots, ℓ - 1, |β_{b,θ}, b and θ as above \} ≤ α,
- \min\{|B_{ℓ,b,θ}| : b ∈ [1, 2], θ ∈ [0, 2π]\} ≥ B_ℓ,
- f(z) = f' (z) = \ldots = f^{(ℓ-1)} (z) = 0 \ (\text{for } f \text{ see (2.7.15)).}

Proof. Let f be a function as in (2.7.14) with unknown numbers α_{b,θ,j}. Then the condition f'(z) = \ldots = f^{(ℓ-1)} (z) = 0 gives the following system of ℓ - 1 equations

\[ \sum_{j=0}^{ℓ-1} s! \left( \frac{2^{N+j+1}}{1 - 2^{j+1}b_θ} \right)^{s+1} α_{b,θ,j} = 0, \ s = 1, \ldots, ℓ - 1, \]

which is equivalent to

\[ \sum_{j=1}^{ℓ-1} s! \left( \frac{2^j}{1 - 2^{j+1}b_θ} \right)^{s+1} α_{b,θ,j} = - \left( \frac{1}{1 - 2b_θ^{1/2}} \right)^{s+1}, \ s = 1, \ldots, ℓ - 1. \quad (2.7.16) \]
Observe that $|A_{b,\theta,j}| \in [1/8, 1]$ and that $A_{b,\theta,\mu} \neq A_{b,\theta,\nu}$ for $\mu \neq \nu$. Now we can rewrite the system of equations (2.7.16) in the following form

$$
\ell - 1 \sum_{j=1}^{\ell-1} A^{s+1}_{b,\theta,j} \alpha_{b,\theta,j} = -A^{s+1}_{b,\theta,0}, \quad s = 1, \ldots, \ell - 1.
$$

From here we conclude that

$$
\left| \det \left( A^{s+1}_{b,\theta,j} \right)_{j,s=1,\ldots,\ell-1} \right| = \left| \prod_{j=1}^{\ell-1} A_{b,\theta,j} \right|^2 \prod_{1 \leq \mu < \nu \leq k} |A_{b,\theta,\mu} - A_{b,\theta,\nu}| \geq \varepsilon > 0,
$$

where $\varepsilon$ is independent of $b$ and $\theta$. Hence the claimed choice of the $\alpha_{b,\theta,j}$, $j = 1, \ldots, \ell - 1$, is always possible. Next, we put

$$
\beta_{b,\theta} := -\sum_{j=0}^{\ell-1} A_{b,\theta,j} \alpha_{b,\theta,j}.
$$

It remains its upper estimate. Observe that

$$
|\beta_{b,\theta}| \leq \sum_{j=0}^{\ell-1} |A_{b,\theta,j} \alpha_{b,\theta,j}| \leq \ell \max\{|\alpha_{b,\theta,j}| : j = 0, \ldots, \ell - 1\}.
$$

Therefore it suffices to estimate the $\alpha_{b,\theta,j}$’s.

Recall that $|A_{b,\theta,j}|^{s+1} \in [2^{-3\ell}, 1]$, $j = 0, \ldots, \ell - 1$, $s = 1, \ldots, \ell - 1$, $b \in [1, 2]$, and $\theta \in [0, 2\pi]$. Applying Cramer’s formula and the continuity of det–function, we see there is a number $\tilde{\alpha} > 0$ such that all the $|\alpha_{b,\theta,j}| \leq \tilde{\alpha}$.

Finally, the lower estimate remains. Since $|B_{\ell,b,\theta}|$ is continuous with respect to $(b, \theta)$ it suffices to show that $B_{\ell,b,\theta} \neq 0$ or equivalently,

$$
\ell - 1 \sum_{j=1}^{\ell-1} A^{s+1}_{b,\theta,j} \alpha_{b,\theta,j} \neq -A^{s+1}_{b,\theta,0}.
$$

Suppose that this is false. Then the $\alpha_{b,\theta,j}$’s fulfill the following $\ell$ equations

$$
\sum_{j=1}^{\ell-1} A^{s+1}_{b,\theta,j} \alpha_{b,\theta,j} = -A^{s+1}_{b,\theta,0}, \quad s = 1, \ldots, \ell,
$$

implying that $A_{b,\theta,0}/A_{b,\theta,j} = 1$, $j = 1, \ldots, \ell - 1$. But this is impossible. Thus also the lower estimate has been proved.

\[ \square \]

2.8. Kobayashi completeness and smoothly bounded pseudoconvex domains

It is well known that there is a bounded pseudoconvex domain $G$ (due to N. Sibony) with a $C^\infty$–boundary except of one point that is not $k_G$–complete (see [J-P 1993], Theorem 7.5.9). On the other hand, for a smoothly bounded pseudoconvex domain $G$ it is still an open question whether it is $k_G$–complete. \[ \square \]
Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and let $z_0 \in \partial D$. Then there is a neighborhood $U = U(z_0)$ and a function $r \in C^\infty(U, \mathbb{R})$ such that

$$D \cap U = \{ z \in U : r(z) < 0 \}$$

and $\text{grad } r(z) \neq 0$, $z \in U$. Moreover, let $\mathcal{V}(z_0)$ be the set of all germs of non-constant holomorphic mappings $\psi : \mathbb{C}_0 \longrightarrow \mathbb{C}^n$ with $\psi(0) = z_0$. According to d’Angelo, the domain $D$ is said to be of finite type at $z_0$ if

$$\tau(D, z_0) := \sup \left\{ \frac{\text{ord}_0 (r \circ \psi)}{\text{ord}_0 \psi} : \psi \in \mathcal{V}(z_0) \right\} < \infty.$$

The domain $D$ is said to be of finite type if $D$ is of finite type at all of its boundary points.

Then there is the following result due to [Bed-For 1978] (see also [For-Sib 1989], [For-McN 1994]).

**Theorem 2.8.1.** Let $D \subset \mathbb{C}^2$ be a bounded pseudoconvex domain. Assume that $D$ is of finite type. Then any boundary point $a \in \partial D$ is a peak point with respect to $O(D) \cap C(U)$, i.e. there exists an $f \in C(U) \cap O(D)$ such that $f(a) = 1$ and $|f(z)| < 1$, $z \in D \setminus \{a\}$.

In particular, we have

**Corollary 2.8.2.** Any bounded pseudoconvex domain with a smooth boundary, which is of finite type, is $c$–complete.

Observe, that to conclude that a domain is $c$–complete, a less strong condition is already sufficient; namely we have (see [J-P 1993]):

**Let $D \subset \mathbb{C}^n$ be a $c$–hyperbolic domain. Then the following two conditions are equivalent :**

(i) $D$ is $c$–finitely compact;
(ii) for any $z_0 \in D$ and for any sequence $(z_j)_j \subset D$ without accumulation points in $D$, there is an $f \in O(D) \cap O(D)$ with $f(z_0) = 0$ and $\sup\{|f(z_j)| : j \in \mathbb{N}\} = 1$.

Up to now it is an open problem whether all bounded pseudoconvex domains of finite type are $k$–complete or even $c$–complete.

**2.9. Kobayashi completeness and unbounded domains**

Let $D \subset \mathbb{C}^n$ be an arbitrary domain and let $a \in \partial D$. The point $a$ is called to be a local holomorphic peak point of $D$, if there is a neighborhood $U = U(a)$ such that $a$ is a peak point with respect to $O(U \cap \overline{D}) \cap O(U \cap D)$. When $D$ is unbounded, we say that $D$ has a local holomorphic peak point at infinity if there is an $r > 0$ and an $f \in C(\overline{D} \setminus \mathbb{B}(0, r), E) \cap O(D \setminus \mathbb{B}(0, r), E)$ such that $\lim_{z \to \infty} f(z) = 1$.

Recall that a bounded domain is locally $k$–complete iff it is $k$–complete (see [J-P 1993], Theorem 7.5.5). For an unbounded domain we have the following result (see [Gau 1999]).

**Theorem 2.9.1.** Let $D \subset \mathbb{C}^n$ be an unbounded domain. Assume that $D$ has a local holomorphic peak point at any point of $\partial D \cup \{\infty\}$. Then $D$ is $k$–complete.
Example 2.9.2. Put

\[ D := \{ z \in \mathbb{C}^2 : u(z) := |z_1|^2(1 + |z_2|^2) < 1 \}. \]

Obviously, \( \{0\} \times \mathbb{C} \subset D \). Thus, \( D \) is not \( k \)-hyperbolic. On the other hand, since \( u \) is strongly psh, any \( a \in \partial D \) is a local holomorphic peak point. So this example shows that the condition at infinity in Theorem 2.9.1 is in some sense necessary.

The proof of Theorem 2.9.1 is based on the following lemma.

Lemma 2.9.3. Let \( D \subset \mathbb{C}^n \) be an arbitrary domain and let \( a \in \overline{\mathbb{C}} \) be a boundary point of \( D \). Assume that \( a \) is a local holomorphic peak point of \( D \). Then:

for any neighborhood \( U = U(a) \) there exists a neighborhood \( V = V(a) \subset U \) such that for any \( \varphi \in \mathcal{O}(E,D), \varphi(0) \in V \), one has \( \varphi(\lambda) \in U, |\lambda| < 1/2 \).

Proof. We give the proof only for \( a = \infty \) (the finite case is similar).

Without loss of generality, let \( U = U(\infty) := \mathbb{C}^n \setminus \overline{B(0,\rho)} \). By assumption, there is an \( r > 0 \) and an \( f \in \mathcal{C}(\overline{D} \setminus \overline{B(0,r)}, E) \cap \mathcal{O}(D \setminus \overline{B(0,r)}, E) \) such that \( \lim_{\partial D z \to \infty} f(z) = 1 \).

We may assume that \( r = \rho \). Put \( u(z) := \log |f(z)|, z \in \overline{D} \setminus \overline{B(0,r)} \). Then

\[ u \in \mathcal{C}(\overline{D} \setminus \overline{B(0,r)}, [-\infty,0)) \cap \mathcal{PSH}(D \setminus \overline{B(0,r)}) \quad \text{and} \quad \lim_{\overline{D} z \to \infty} u(z) = 0. \]

Fix numbers \( r', r'', r < r' < r'' \), such that

\[ \sup \{ u(z) : z \in D \cap \partial \overline{B}(0,r') \} =: c' < 0, \]
\[ \inf \{ u(z) : z \in D \cap \partial \overline{B}(0,r'') \} =: c'' > c', \quad \text{and} \quad f(z) \neq 0, \|z\| \geq r'. \]

We define \( \hat{u} : D \to (-\infty,0) \),

\[ \hat{u}(z) := \begin{cases} u(z), & \text{if } \|z\| \geq r'' \\ \max \{ u(z), \frac{c' + c''}{2} \}, & \text{if } r' < \|z\| < r'' \\ \frac{c' + c''}{2}, & \text{if } \|z\| \leq r' \end{cases}. \]

Obviously, \( \hat{u} \) is a global negative psh peak function at \( \infty \), i.e. \( \hat{u} \) is a negative continuous function on \( D \), psh on \( D \), such that \( \lim_{\partial D z \to \infty} \hat{u}(z) = 0 \).

Fix an \( \psi \in \mathcal{C}(\overline{E}, D) \cap \mathcal{O}(E, D) \). Observe that \( \hat{u} \circ \psi \in \mathcal{C}(\overline{E}) \cap \mathcal{SH}(E) \).

Let \( \alpha < 0 \). Put

\[ E(\psi, \alpha) := \{ \theta \in [0,2\pi] : \hat{u} \circ \psi(e^{i\theta}) \geq 2\alpha \}. \]

Assume that \( \alpha \leq \hat{u} \circ \psi(0) \). Then

\[ \alpha \leq \hat{u} \circ \psi(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \hat{u} \circ \psi(e^{i\theta}) d\theta \]
\[ \leq \frac{1}{2\pi} \int_{[0,2\pi] \setminus E(\psi,\alpha)} \hat{u} \circ \psi(e^{i\theta}) d\theta \leq \frac{\alpha}{\pi} (2\pi - A_1(\hat{u} \circ \psi(\alpha))). \]

Hence,

\[ A_1(\hat{u} \circ \psi(\alpha)) \geq \pi. \] (2.9.17)
Put \( v(z) := \log \frac{|f(z) - 1|}{2} \), \( z \in \overline{D} \). Then \( v \in C(\overline{D} \setminus B(0, r)) \cap \mathcal{P}SH(D \setminus \overline{B}(0, r)) \). Choose an \( \varepsilon > 0 \) such that

\[
\sup \{(u + \varepsilon v)(z) : z \in \overline{D} \cap \partial B(0, r')\} =: c'_1 < 0,
\]

\[
\inf \{(u + \varepsilon v)(z) : z \in \overline{D} \cap \partial B(0, r'')\} =: c''_1 > c'_1.
\]

Define \( \hat{\nu} : \overline{D} \rightarrow (-\infty, 0) \) as

\[
\hat{\nu}(z) := \begin{cases} 
(u + \varepsilon v)(z), & \text{if } ||z|| \geq r'' \\
\max\{(u + \varepsilon v)(z), \frac{c'_1 + c''_1}{2}\}, & \text{if } r' < ||z|| < r'' .
\end{cases}
\]

Then \( \hat{\nu} \in C(\overline{D}) \cap \mathcal{P}SH(D) \), \( \hat{\nu} < 0 \), and \( \lim_{D \to \infty} \hat{\nu}(z) = -\infty \). (Such a function is sometimes called a global psh antipeak function at \( \infty \).

Let \( \psi \) be as above. Applying the Poisson integral representation, we get

\[
\hat{\nu} \circ \psi(\lambda) \leq \frac{1}{2\pi} \int_0^{2\pi} \hat{\nu} \circ \psi(e^{i\theta}) \frac{1 - |\lambda|^2}{|e^{i\theta} - \lambda|^2} d\theta \leq \frac{1}{6\pi} \int_0^{2\pi} \hat{\nu} \circ \psi(e^{i\theta}) d\theta, \quad |\lambda| \leq \frac{1}{2} \quad (2.9.18)
\]

Now, choose \( L > 0 \) such that

\[
U' := \{z \in \overline{D} : \hat{\nu}(z) < -\frac{L}{6}\} \subset \mathbb{C}^n \setminus \overline{B}(0, r).
\]

Then there is an \( \alpha_0 > 0 \) such that

\[
V := \{z \in \overline{D} : \hat{\nu}(z) > -\alpha_0\} \subset \{z \in \overline{D} : \hat{\nu}(z) \geq -2\alpha_0\} \subset \{z \in \overline{D} : \hat{\nu}(z) < -L\}.
\]

Now let \( \varphi \in \mathcal{O}(E, D) \) such that \( \varphi(0) \in V \). Obviously, we may also assume that \( \varphi \in C(\overline{E}, D) \).

Applying (2.9.17) and (2.9.18) we have for \( \lambda \in E, |\lambda| \leq 1/2:\)

\[
\hat{\nu} \circ \varphi(\lambda) \leq \frac{1}{6\pi} \int_0^{2\pi} \hat{\nu} \circ \varphi(e^{i\theta}) d\theta \leq \frac{1}{6\pi} \int_{E(\varphi, \alpha_0)} \hat{\nu} \circ \varphi(e^{i\theta}) d\theta \leq -\frac{LA_1(E(\varphi, \alpha_0))}{6\pi} \leq -\frac{L}{6},
\]

i.e. \( \varphi(\lambda) \in U' \).

**Corollary 2.9.4.** Let \( D \subset \mathbb{C}^n \) and a be as in Lemma 2.9.3. Let \( U = U(a) \) be any neighborhood of \( a \). Then there exists a neighborhood \( V = V(a) \subset U \) such that for any connected component \( V' \) of \( D \cap V \) the following inequality is true:

\[
2\kappa_D(z; X) \geq \kappa_{V}(z; X), \quad z \in V', X \in \mathbb{C}^n.
\]

**Remark 2.9.5.** Observe that in Lemma 2.9.3 only the existence of a local psh peak function and a local psh antipeak function was needed. Other localization results for unbounded domains may be found in [Nik 2002].

**Proof of Theorem 2.9.1.** Step 1. We prove that \( D \) is \( k \)-hyperbolic if \( D \) has a local holomorphic peak point at infinity.

Assume this is not the case. Then there exist \( z_0 \in D \), \( (z_j)_j \subset D \) with \( z_j \rightarrow z_0 \), and \( X_j \in \mathbb{C}^n \) with \( ||X_j|| = 1 \) such that \( \kappa_D(z_j; X_j) < 1/j, j \in \mathbb{N} \) (see Theorem 7.2.2 in [J-P 1993]). Therefore, we find functions \( \varphi_j \in \mathcal{O}(E, D) \) such that \( \varphi_j(0) = z_j \) and...
Moreover, put \( U \). According to Corollary 2.9.4, we find an \( R > 0 \) and a \( \varphi \) no local holomorphic peak function at infinity.

The following domain (see [Par 2003])

Remark 2.9.6. contradiction. \( \Box \)

Fix an \( a \) and let \( \{z, \lambda\} \in D \times C : |\lambda| < e^{-u_a(z)} \} \).

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\( \|\varphi_j(0)\| > j, j \in \mathbb{N} \). In virtue of the Cauchy inequalities, we may further assume that there is a sequence \((\lambda_j)_j \in \frac{1}{2}E, \lambda_j \to 0\), such that \( \|\varphi_j(\lambda_j)\| \to \infty \).

Put \( \tilde{\varphi}_j := \varphi_j \circ (-h_{\lambda_j}) \). Then \( \tilde{\varphi}_j \in O(E, D) \) with \( \tilde{\varphi}_j(\lambda_j) = z_j \) and \( \|\tilde{\varphi}_j(0)\| \to \infty \).

Put \( R := 2\|z_0\| + 1 \). Then there is an \( R' > R \) for which Lemma 2.9.3 can be used. Since \( \|\tilde{\varphi}_j(0)\| > R' \) for large \( j \), we get for these \( j \) that \( \|\tilde{\varphi}_j(\lambda_j)\| > R \), which contradicts the fact that \( z_j \to z_0 \).

Step 2. Here we prove that \( D \) is \( k \)-complete. Assume the contrary. Then there are a point \( z_0 \in D \) and a sequence \((z_j)_j \subset D \) such that \( A := \sup\{k_D(z_0, z_j) : j \in \mathbb{N}\} < \infty \) and either \( z_j \to z^* \in \partial D \) or \( z_j \to \infty \). Again we discuss only the second case. The first is similar.

Let \( f \in C(\overline{D} \setminus \mathbb{B}(0, r), E) \cap O(D \setminus \overline{\mathbb{B}(0, r)}, E) \) be the local holomorphic peak function at infinity.

Choose \( C^1 \)-curves \( \alpha_j : [0, 1] \to D \) with

\[
\alpha_j(0) = z_0, \quad \alpha_j(1) = z_j, \quad \int_0^1 \kappa_D(\alpha_j(t); \alpha'_j(t))dt < A + 1.
\]

According to Corollary 2.9.4, we find an \( R > \max\{\|z_0\|, r\} \) such that for any connected component \( U \) of \( D \cap (C^n \setminus \overline{\mathbb{B}(0, R)}) \) the following is true: \( 2\kappa_D(z; X) \geq \kappa_U(z; X), z \in U, X \in C^n \). Moreover, observe that \( |f| \leq C < 1 \) on \( \overline{D} \setminus \partial \mathbb{B}(0, R) \).

We may assume that all \( \|z_j\| > R \). Fix an \( j \), put \( t_j := \sup\{t \in [0, 1] : \|\alpha_j(t)\| \leq R\} \), and let \( U_j \) denote that connected component of \((C^n \setminus \overline{\mathbb{B}(0, R)}) \cap D\) containing \( \alpha_j(t_j, 1) \). Then

\[
2\int_0^1 \kappa_D(\alpha_j(t); \alpha'_j(t))dt \geq \int_{t_j}^1 \kappa_{U_j}(\alpha_j(t); \alpha'_j(t))dt \geq \int_{t_j}^1 \kappa_E(f \circ \alpha_j(t); (f \circ \alpha_j)'(t))dt \geq \min\{k_E(f(z_j), \lambda) : |\lambda| \leq C\} \to \infty;\]

contradiction. \( \Box \)

Remark 2.9.6. The following domain (see [Par 2003])

\[
D := \{(z, w) \in C^3 \times C : |z_1z_2z_3| < 1, 0 < |w| < e^{-\max\{|z_j|; j=1,2,3\}}\}
\]

is \( k \)-complete, but there is no local psh peak function at infinity; in particular, there is no local holomorphic peak function at infinity.

Indeed, \( D \) is a pseudoconvex Reinhardt domain which is Brody–hyperbolic. Hence, it is \( k \)-complete (see Theorem 2.2.1).

Assume now that there exists a local psh peak function at \( \infty \). Hence there is an \( R > 1 \) and a \( \varphi \in C(\overline{D} \setminus \overline{\mathbb{B}(0, R)}) \cap \mathcal{PSH}(D \setminus \overline{\mathbb{B}(0, R)}), \varphi < 0 \), such that \( \lim_{\overline{D}(z,w) \to \infty} \varphi(z, w) = 0 \).

Fix an \( a \in C \) with \( |a| = 2R \) and define

\[
D_a := \{z \in C^2 : 2R|z_1z_2| < 1\} \text{ and } u_a(z) := \max\{|z_1|, |z_2|, |a|\}, \quad z \in D_a.
\]

Moreover, put

\[
\Omega := \{(z, \lambda) \in D_a \times C : |\lambda| < e^{-u_a(z)}\}.
\]
Finally, set \( \varphi_a : \Omega \rightarrow [-\infty, \infty) \) as \( \varphi_a(z, \lambda) := \varphi(z, a, \lambda) \). Then \( \varphi_a \in \mathcal{PSH}(\Omega) \) and \( \varphi_a(z, 0) \in \mathcal{PSH}(D_a) \). In virtue of the Liouville theorem, there is a \( \psi \in \mathcal{SH}(\frac{1}{2\pi}E) \) such that \( \varphi_a(z, 0) = \psi(z_1 z_2) \), \( z \in D_a \). So, applying the Oka theorem, we get

\[
\psi(0) = \limsup_{R \to \infty} \psi\left(\frac{1}{2Rt}\right) = \limsup_{t \to \infty} \varphi_a\left(t, \frac{1}{2Rt^2}, 0\right) =: C \leq 0.
\]

In the case when \( C=0 \), the maximum principle would imply that \( \psi \equiv 0 \) on \( \frac{1}{2\pi}E \) and thus \( \varphi(\cdot, a, 0) \equiv 0 \) on \( D_a \) which contradicts the assumption \( \varphi < 0 \). Hence, \( C < 0 \).

Now choose a \( t_0 > 2|a| \) such that for all \( t > t_0 \) the following inequality is true:

\[
\lim_{0 \neq \lambda \to 0} \varphi\left(t, \frac{1}{2Rt^2}, a, \lambda\right) = \varphi_a\left(t, \frac{1}{2Rt^2}, 0\right) < \frac{3}{4} C.
\]

So, for \( t > t_0 \), there is an \( \varepsilon_t > t \) such that

\[
0 < C - \varepsilon_t < \frac{1}{2} C, \quad 0 < |\lambda| < e^{-\varepsilon_t}.
\]

Now observe that \( \max\{t, \frac{1}{2Rt}, |a|\} = t, t > t_0, \) and \( \varepsilon_t \to \infty \) if \( t \to \infty \). Therefore, we may choose a sequence \( \{(t_j, \frac{1}{2Rt_j}, a, \lambda_j)\} \subset D \cap (\mathbb{R}^2 \times \{a\} \times \mathbb{C}) \) such that \( t_0 < t_j \to \infty \) and \( |\lambda_j| < e^{-\varepsilon_j} \). Hence, we have

\[
0 = \lim_{j \to \infty} \varphi(t_j, \frac{1}{2Rt_j^2}, a, \lambda_j) \leq \frac{C}{4};
\]

a contradiction.

The above example shows that the conditions in Theorem 2.9.1 are too strong. Observe that any bounded boundary point \( z_0 \) is obviously a local psh antipeak point (take simply \( \log \|z-z_0\| \) for a large \( R \)).

Does Theorem 2.9.1 remain true if one only assumes that any boundary point admits a local psh peak and antipeak function? 

In this context observe that there exists a smoothly bounded pseudoconvex domain \( D \subset \mathbb{C}^3 \) such that each boundary point of \( D \) is a global psh peak point, but some boundary point is not a local holomorphic peak point (see [Yu 1997]).

**Example 2.9.7.** Theorem 2.9.1 has been used in [Gau 1999] to prove the following results.

(a) Let \( P \) be a real valued subharmonic polynomial on \( \mathbb{C} \) without harmonic terms. Then

\[
D := \{(z, w) \in \mathbb{C} \times \mathbb{C} : \Re w + P(z) < 0\}
\]

is \( k \)-complete.

(b) Let \( P \) be a real valued convex polynomial on \( \mathbb{C}^n, \ P(0) = \text{grad} \ P(0) = 0 \), without harmonic terms, such that the set \( \{z \in \mathbb{C}^n : P(z) = 0\} \) does not contain a nontrivial analytic set. Then

\[
D := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \Re w + P(z) < 0\}
\]
is a convex $k$–complete domain.\footnote{See also Theorem 7.1.8 in [J-P 1993] for the following characterization of $k$–complete convex domains: A convex domain $G$ is $k$–complete iff $G$ contains no complex lines iff $G$ is biholomorphic to a bounded convex domain.}
CHAPTER 3

Bergman metric

3.1. The Bergman kernel

In this chapter we will discuss a metric on domains which is invariant under biholomorphic mappings, namely the Bergman metric. To do so we have to recall first the Bergman kernel function and the Bergman kernel.

Let $G \subset \mathbb{C}^n$ be a domain. We denote by $L^2_h(G)$ the Hilbert space of all square integrable functions on $G$ which are holomorphic; it is a closed subspace of $L^2(G)$. The key tool in this chapter is the following extension theorem due to K. Ohsawa and K. Takegoshi [Ohs-Tak 1987].

Theorem 3.1.1. Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $H$ an affine subspace of $\mathbb{C}^n$. Then there is a positive constant $C$, which depends only on the diameter of $D$ and $n$, such that for any $f \in L^2_h(D \cap H)$ there is an $F \in L^2_h(D)$ such that $F|_{D \cap H} = f$ and $\|F\|_{L^2_h(D)} \leq C\|f\|_{L^2_h(D \cap H)}$.

Moreover, we recall the following one-dimensional result (see [Lin 1977], [Che 2000]) which will be used in the sequel.

Theorem 3.1.2. Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in \partial D$, and $f \in L^2_h(D)$. Then for any $\varepsilon > 0$ there exist a neighborhood $U = U(z_0)$ and a function $g \in L^2_h(D \cup U)$ such that $\|f - g\|_{L^2_h(D)} \leq \varepsilon$. In particular, the subspace of all functions in $L^2_h(D)$, bounded near $z_0$, is dense in $L^2_h(D)$.

In [Che 2000], complete Kaehler metrics were used to solve a corresponding $\overline{\partial}$-problem in order to find $g$. Here we give a proof which is based on Berndtsson’s solution of a $\overline{\partial}$-problem (see [Pfl 2000]).

Proof. We may assume that $z_0 = 0 \in \partial D$ and that $\overline{D} \subset E$. Fix $f \in L^2_h(D)$ and a sufficiently small $\varepsilon \in (0, 1/2)$. Put $\psi(z) := -\log(\log(1/|z|)), z \neq 0$. Observe that $\psi \in C^\infty(C^*_+) \cap SH(C^*_+)$ and $|\partial\psi|_0^2 = |\overline{\partial}\psi| = \log(1/|z|^2) - 2 > 0$.

Moreover, let $\chi \in C^\infty([0, 1])$, $\chi(t) := \begin{cases} 1, & \text{if } t \leq 1 - \log 2 \\ 0, & \text{if } t > 1 \end{cases}$, be such that $|\chi'| \leq 3$.

Finally, we define $\rho_\varepsilon(z) := \chi(-\psi(z) - \log(\log(1/\varepsilon)) + 1), z \in \mathbb{C}_+$. Observe that $\rho_\varepsilon(z) = 0$ if $0 < |z| < \varepsilon$, and $\rho_\varepsilon(z) = 1$ if $|z| > \sqrt{\varepsilon}$. 

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Then \( \alpha := \overline{\partial}(\rho_{z}f) \) is a \( C^{\infty} \) \( \overline{\partial} \)-closed \((0,1)\)-form on \( D_{\varepsilon} := D \cup B(0, \varepsilon) \). Now we like to apply the following theorem of Berndtsson.

**Theorem** ([Ber 1996]). Let \( \Omega \subset \mathbb{C}^{n} \) be a bounded pseudoconvex domain. Let \( \varphi, \psi \in \mathcal{PSH}(\Omega) \), \( \psi \) strongly psh, be such that for any \( X \in \mathbb{C}^{n} \) the following inequality

\[
\sum_{j,k=1}^{n} \frac{\partial^{2} \psi}{\partial z_{j} \partial \overline{z}_{k}}(z)X_{j}X_{k} \geq \sum_{j=1}^{n} \left| \frac{\partial \psi}{\partial z_{j}}(z) \right|^{2}
\]

holds on \( \Omega \). Let \( \delta \in (0, 1) \) and \( \alpha = \sum_{j=1}^{n} \alpha_{j} d\overline{z}_{j} \) a \( \overline{\partial} \)-closed \((0,1)\)-form.

Then there exists a solution \( u \in L^{2}_{\text{loc}}(\Omega) \) of \( \overline{\partial}u = \alpha \) such that

\[
\int_{\Omega} \left| u \right|^{2} e^{-\varphi + \delta \psi} dA_{2n}(z) \leq \frac{4}{\delta(1-\delta)^{2}} \int_{\Omega} \sum_{j,k=1}^{n} \psi^{j,k} \alpha_{j} \overline{\alpha_{k}} e^{-\varphi + \delta \psi} dA_{2n}(z),
\]

where \( (\psi^{j,k}) \) denotes the inverse matrix of \( (\frac{\partial^{2} \psi}{\partial z_{j} \partial \overline{z}_{k}}) \).

Take \( \varphi := (1/2) \psi \) and \( \delta := 1/2 \). Then there exists a function \( u_{\varepsilon} \), \( \overline{\partial}u_{\varepsilon} = \alpha \) on \( D_{\varepsilon} \setminus \{0\} \) such that

\[
\int_{D_{\varepsilon} \setminus \{0\}} \left| u_{\varepsilon} \right|^{2} e^{-\varphi + \delta \psi} dA_{2n}(z) \leq \frac{4}{\delta(1-\delta)^{2}} \int_{D_{\varepsilon} \setminus \{0\}} \left| \alpha \right|^{2} e^{-\varphi + \delta \psi} dA_{2n}(z) = 16 \int_{z \in D, \varepsilon \leq |z| \leq \sqrt{\varepsilon}} |\chi|^{2} \cdot |f|^{2} dA_{2}(z).
\]

Then the function \( f_{\varepsilon} := u_{\varepsilon} - \rho_{z}f \) belongs to \( L^{2}_{D}(D \setminus \{0\}) \) and

\[
\|f - f_{\varepsilon}\|_{L^{2}_{D}(\Omega)} \leq \|(1 - \rho_{z})f\|_{L^{2}_{D}(\Omega)} + 160 \|\alpha\|_{L^{2}_{D}(\Omega \times \mathbb{R}(0, \sqrt{\varepsilon}))} \leq C \|f\|_{L^{2}_{D}(\Omega \times \mathbb{R}(0, \sqrt{\varepsilon}))} \xrightarrow{\varepsilon \to 0} 0,
\]

where \( C \) is a general positive constant. It remains to note that \( f_{\varepsilon} \in \mathcal{O}(D_{\varepsilon}) \) (use Laurent series), which finishes the proof. \( \square \)

We note that under some proper assumptions this result can be generalized to higher dimensions.

Observe that the point evaluation functional \( L^{2}_{p}(G) \ni f \longrightarrow f(w) \ (w \in G) \) is continuous. Therefore, there is a uniquely defined function \( K_{G}(\cdot, w) \in L^{2}_{p}(G) \) such that

\[
f(w) = \int_{G} \overline{f(z)K_{G}(z, w)} dA(z), \quad f \in L^{2}_{p}(G), w \in G.
\]

The function \( K_{G} \) is the **Bergman kernel function for** \( G \). Recall that \( K_{G} \) can be given with the help of a complete orthonormal system \( \{\varphi_{j}\}_{j \in N} \subset L^{2}_{p} \), namely

\[
K_{G}(z, w) = \sum_{j \in N} \varphi_{j}(z)\overline{\varphi_{j}(w)}, \quad z, w \in G.
\]

**Remark 3.1.3.** Recall that there are domains \( G_{k} \subset \mathbb{C}^{2} \), for which \( \dim L^{2}_{p}(G_{k}) = k \) [Wig 1984]. \( \square \) It is unknown whether \( \dim L^{2}_{p}(G) = \infty \), if \( G \subset \mathbb{C}^{n} \), \( n > 1 \), is a pseudoconvex domain with \( L^{2}_{p}(G) \neq \{0\} \). \( \square \)
The function $K_G$ is holomorphic in $z$ and antiholomorphic in $w$; moreover, we have $K_G(z, w) = \overline{K_G(w, z)}$, $z, w \in G$. If $\Phi : G \to D$ is a biholomorphic mapping between the domains $D$ and $G$, then
\[
K_D(\Phi(z), \Phi(w)) \det \Phi'(z) \overline{\det \Phi'(w)} = K_G(z, w), \quad z, w \in G.
\]
Moreover, there is a transformation law even for proper holomorphic mappings due to S. Bell (see [J-P 1993], Theorem 6.1.8).

**Theorem 3.1.4.** Let $F : G \to D$ be a proper holomorphic mapping of order $m$ between the bounded domains $G, D \subset \mathbb{C}^n$. Let $u := \det F'$ and denote by $\Phi_1, \ldots, \Phi_m$ the local inverses of $F$ defined on $D' := D \setminus \{F(z) : z \in G, u(z) = 0\}$. Put $U_k := \det \Phi_k'$. Then
\[
m \sum_{k=1}^m K_G(z, \Phi_k(w)) U_k(w) = u(z) K_D(F(z), w), \quad z \in G, \ w \in D'.
\]

The function $k_G(z) := K_G(z, z)$ is called the Bergman kernel of $G$. In the case when $L^2_h(G) \neq \{0\}$, then $k_G$ is also given as
\[
k_G(z) = \sup \left\{ \frac{|f(z)|^2}{\|f\|^2_{L^2_h}} : f \in L^2_h(G) \setminus \{0\} \right\}.
\]

Observe that $k_D|G \leq k_G$ whenever $G \subset D$.

For the Bergman kernel there is the following localization result (see [J-P 1993], Theorem 6.3.5).

**Theorem 3.1.5.** Let $D_1 \subset \mathbb{C}^n$, $j = 1, 2$, be two bounded pseudoconvex domains and let $z_0 \in \partial D_j$. Assume that there is a neighborhood $U = U(z_0)$ of $z_0$ such that $D_1 \cap U = D_2 \cap U$. Then there exist positive numbers $m, M$ and a neighborhood $V = V(z_0)$ such that
\[
m k_{D_1}(z) \leq k_{D_2}(z) \leq Mk_{D_1}(z), \quad z \in V \cap D_1.
\]

In general, it is not easy to find explicit formulas for the Bergman kernel function. In most of the known examples the formulas are obtained using an explicit complete orthonormal system $(\varphi_j) \subset L^2_h(G)$.

**Example 3.1.6** (see Examples 6.1.5 and 6.1.6 in [J-P 1993]). (a) For the Euclidean ball $B_n$ we have
\[
K_{B_n}(z, w) = \frac{n!}{\pi^n} \left(1 - \langle z, w \rangle \right)^{-(n+1)}, \quad z, w \in B_n.
\]
(b) Let $E^n$ be the $n$–dimensional polydisc. Then
\[
K_{E^n}(z, w) = \frac{1}{\pi^n} \prod_{j=1}^n \left(1 - z_j \overline{w_j} \right)^{-2}, \quad z, w \in E^n.
\]

\footnote{Observe that the symbol $k_D(\cdot)$ is a function on $D$, while the Kobayashi pseudodistance $k_D(\cdot, \cdot)$ is defined on $D \times D$; we hope there will be no confusion for the reader.}
Example 3.1.7 as an open question is whether $K_k$ (where also \cite{Boa 2000}). Calculation.

It is easily seen that (d) Recently, using Theorem 3.1.4, the following formula for the Bergman kernel (c) Put $D_p := \{ z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2/p} < 1 \}$, $p > 0$. Then

$$K_{D_p}(z, w) = \frac{1}{\pi^2} (1 - z_1 \overline{w}_1)^{p-2} \frac{(p+1)(1 - z_1 \overline{w}_1)^p + (p-1)z_2 \overline{w}_2}{(1 - z_1 \overline{w}_1)^p - z_2 \overline{w}_2}^3, \quad z, w \in D_p.$$  

Observe (by a simple calculation) that $K_{D_2}$ has no zeros on $D_2 \times D_2$.

(d) Recently, using Theorem 3.1.4, the following formula for the Bergman kernel function of $G_n$ has been found in \cite{Edi-Zwo 2004} (see Remark 1.4.17 for a definition of $G_n$).

$$K_{G_n}(\pi_n(z), \pi_n(w)) = \frac{\det \left[ \frac{1}{(1 - \lambda_j \overline{\mu}_k)^{p-2}} \right]_{1 \leq j, k \leq n}}{\pi^n \det \pi'_n(\lambda) \det \pi'_n(\mu)} = \frac{F_n(z, \overline{w})}{\pi^n \prod_{j, k=1}^n (1 - \lambda_j \overline{\mu}_k)^2}, \quad \lambda, \mu \in E^n \setminus \{ \zeta \in E^n : \det \pi'_n(\zeta) = 0 \} = E^n \setminus \{ \zeta \in E^n : \zeta_j = \zeta_k \text{ for some } j \neq k \},$$

where $\pi_n : \mathbb{C}^n \to \mathbb{C}^n$,

$$\pi_n(\lambda_1, \ldots, \lambda_n) := \left( \sum_{1 \leq j_1 < \cdots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k} \right)_{k=1, \ldots, n} \quad (G_n = \pi_n(E^n)).$$

In particular,

$$K_{G_2}(\pi_2(\lambda), \pi_2(\mu)) = K_{G_2}(z, w) = \frac{F_2(z, \overline{w})}{\pi^2 \prod_{j, k=1}^2 (1 - \lambda_j \overline{\mu}_k)^2}, \quad \lambda, \mu \in E^2,$$

where

$$F_2(z, w) := 2 - z_1 w_1 + 2z_2 w_2,$$

$$K_{G_3}(\pi_3(\lambda), \pi_3(\mu)) = K_{G_3}(z, w) = \frac{F_3(z, \overline{w})}{\pi^3 \prod_{j, k=1}^3 (1 - \lambda_j \overline{\mu}_k)^2}, \quad \lambda, \mu \in E^3,$$

where

$$F_3(z, w) := 6 - 4z_1 w_1 - 2z_2 w_2 + 2z_1^2 w_2 + 2z_2 w_1^2 - 3z_1 z_2 w_3 - 3z_1 w_1 w_2 + 15z_3 w_3 - z_1 z_2 w_1 w_2 - 2z_1 z_3 w_1 w_3 + 2z_1 z_2 w_2 w_3 - 2z_1 z_2 w_1 w_3 - 4z_2 z_3 w_2 w_3 + 6z_2^2 w_2^3.$$  

It is easily seen that $K_{G_2}$ has no zeros on $G_2 \times G_2$ — use simply the description of $\text{Aut}(G_2)$ (Theorem 1.4.14) to reduce the discussion to the case $\mu_2 = 0$. What remains as an open question is whether $K_{G_n}$ with $n \geq 3$ has zeros.

Example 3.1.7 (\cite{D’Ang 1994}). In generalizing Example 3.1.6(c), let

$$D := \{ \zeta = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|z\|^2 + \|w\|^{2p} < 1 \}$$

with $p \in (0, \infty)$. Then we have the following formula for its Bergman kernel:

$$k_D(\zeta) = \sum_{j=0}^{n+1} c_k \frac{(1 - \|z\|^2)^{-n-1+k_j}}{(1 - \|z\|^2)^{\frac{k_j}{2}} - \|w\|^2}^{m+k_j}, \quad \zeta = (z, w) \in D,$$

where the constants $c_k$ depend on $k$, $n$, $m$, and $p$. Even more, they can be explicitly calculated.

Other examples and methods how to proceed may be found in \cite{Boa-Fu-Str 1999} (see also \cite{Boa 2000}).
3.1.1. Deflation. Fix a bounded domain $D \subset \mathbb{C}^n$ which is given as

$$D = \{ z \in U : \varphi(z) < 1 \},$$

where $\varphi \in \mathcal{C}(U, [0, \infty))$ for a suitable open neighborhood $U$ of $\overline{D}$. Put

$$G_1 := \{ (z, \zeta) \in D \times \mathbb{C}^1 : \varphi(z) + |\zeta|^{2/(p+q)} < 1 \},$$

$$G_2 := \{ (z, \zeta) \in D \times \mathbb{C}^2 : \varphi(z) + |\zeta_1|^{2/p} + |\zeta_2|^{2/q} < 1 \},$$

where $p, q$ are positive real numbers. Then we have the following deflation identity

$$\pi K_{G_1}((z, 0), (w, 0)) = \frac{\pi^2 \Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)} K_{G_2}((z, 0), (w, 0)), \quad z, w \in D. \quad (D)$$

In fact, the identity $(D)$ holds because both sides represent the unique reproducing kernel for the Hilbert space $L^2_h(D, \pi(1-\varphi)^{p+q})$ \footnote{Recall that, if $\psi$ is a non-negative measurable function, then $L^2_h(D, \psi) := \{ f \in \mathcal{O}(D) : \int_D |f|^2 \psi dA_{2n} < \infty \}$.}. To be more precise, fix an $h \in L^2_h(D)$. Then $h$ can be also thought to belong to $L^2_h(G_j)$, $j = 1, 2$. Therefore, in virtue of the reproducing property of the Bergman kernel function, we see that

$$h(z) = \int_{G_1} h(w)K_{G_1}((z, 0), (w, \zeta)) \, d\Lambda(w, \zeta).$$

Observe that the fiber over a point $w \in D$ is a disc of radius $(1 - \varphi(w))^{(p+q)/2}$. Therefore, applying the mean value property for harmonic functions leads to

$$h(z) = \int_D h(w)(1 - \varphi(w))^{p+q} \pi K_{G_1}(z, 0), (w, 0)) \, d\Lambda(w).$$

Hence, $K_{G_1}((0, 0), (0, 0))$ is the reproducing kernel for $L^2_h(D, \pi(1-\varphi)^{p+q})$.

A similar reasoning leads to the same conclusion for the right side of $(D)$, which finally proves the deflation identity.

**Example 3.1.8.** For example, let $G := \{ z \in \mathbb{C}^2 : |z_1| + |z_2|^{1/2} < 1 \}$. Assume we knew the formula for $K_G(z, w)$ (see Example 3.2.1). Now, let $D = E$ and $p = q = 2$. Applying the deflation method from above, we get for $G^* := \{ z \in \mathbb{C}^3 : |z_1| + |z_2| + |z_3| < 1 \}$

$$\pi K_G((z, 0), (w, 0)) = \frac{\pi^2}{3!} K_{G^*}((z, 0, 0), (w, 0, 0)), \quad z, w \in E.$$

Observe that if $K_G((z, 0), (w, 0)) = 0$ for certain points $z, w \in E$, then we also have $K_{G^*}((z, 0, 0), (w, 0, 0)) = 0$.

Most of the domains for which an explicit formula for its Bergman kernel is known are Reinhardt domains. Here we describe the Bergman kernel function of a domain that is not biholomorphically equivalent to a Reinhardt domain. Define

$$N(z) := \sqrt{\|z\|^2 + |z \cdot z|}, \quad z \in \mathbb{C}^n.$$
Recall that \( N/\sqrt{2} \) is the smallest \( \mathbb{C} \)-norm dominated by \( \| \cdot \| \) which coincides with the Euclidean norm on \( \mathbb{R}^n \) (cf. [Hah-Pfl 1988]). The \( N \)-ball \( M_n \) is called the *minimal ball*, i.e.

\[
M := M_n := \{ z \in \mathbb{C}^n : N(z) < 1 \}.
\]

Observe that \( M_n \) can be thought as a model for domains with non smooth boundary.

For \( n = 2 \), a formula for its Bergman kernel function can be found in [J-P 1993]. The general case is contained in [Oel-Pfl-You 1997] and [Men-You 1999].

**Theorem 3.1.9.** The Bergman kernel function of \( M \) is given by the following formula

\[
K_M(z, w) = \frac{1}{n(n + 1)A_{2n}(M)} \sum_{j=0}^{[\hat{T}]} \frac{(n+1)!X^{n+1-2j}Y^j(2nX - (n - 2j)(X^2 - Y))}{(X^2 - Y)^{n+1}},
\]

where \( z, w \in M, X = X(z, w) := 1 - \langle z, w \rangle, \) and \( Y = Y(z, w) := (z \bullet z)(w \bullet w) \).

**Proof.** The main ideas of the proof are:

1) to establish a formula for the Bergman kernel function of the “domain”

\[
\mathcal{R} := \{ z \in \mathbb{C}^{n+1} \setminus \{ 0 \} : \| z \| < 1, z \bullet z = 0 \},
\]

2) to use the proper mapping \( \pi : \mathcal{R} \to M \setminus \{ 0 \}, \pi(\tilde{z}, z_{n+1}) := \tilde{z}, \) to get a formula for the Bergman kernel function of \( M \).

Now, we present the proof in more details. First, observe that the following \( n \)-form on \( \mathbb{C}^{n+1} \)

\[
\hat{\alpha}(z) := \sum_{j=1}^{n+1} \frac{(-1)^{j+1}z_j}{\hat{z}_j} dz_1 \wedge \cdots \wedge \hat{z}_j \wedge \cdots \wedge dz_{n+1}
\]

induces by restriction an \( SO(n + 1) \)-invariant holomorphic \( n \)-form \( \alpha \) on the complex manifold \( \mathcal{H} := \{ z \in \mathbb{C}^{n+1} \setminus \{ 0 \} : z \bullet z = 0 \} \).

Put

\[
\omega(z)(V_1, \ldots, V_{n-1}) := \alpha(z) \wedge \overline{\alpha(z)}(z, V_1, \ldots, V_{n-1}), \quad z \in \mathcal{R},
\]

where \( (V_1, \ldots, V_{n-1}) \in T_z(\partial \mathcal{R}) \). Observe that \( \omega \) is a volume form on \( T_z(\partial \mathcal{R}) \). Since \( \alpha \wedge \overline{\alpha} \) is \( SO(n + 1) \)-invariant, also \( \omega \) is \( SO(n + 1) \)-invariant. Hence the measure on \( \partial \mathcal{R} \) induced by \( \omega \) is proportional to the unique \( O(n + 1, \mathbb{R}) \)-invariant measure \( \mu \) on \( \partial \mathcal{R} \) with \( \mu(\partial \mathcal{R}) = 1 \). Put \( \omega(\partial \mathcal{R}) := \int_{\partial \mathcal{R}} \omega \).

Exploiting the definition of the form \( \alpha \), the following statement may be proved:

For any \( C^\infty \)-function \( f \) on \( \mathcal{H} \) we have

\[
\int_{\partial \mathcal{H}} f(z) \alpha(z) \wedge \overline{\alpha(z)} = \omega(\partial \mathcal{R}) \int_0^\infty \int_{\partial \mathcal{R}} f(t\zeta)d\mu(\zeta)dt,
\]

provided the integrals make sense.

Moreover, using harmonic sphericals, the following result is true:

\[
\int_{\partial \mathcal{R}} (z \bullet w)^k (\xi \bullet \overline{w})^k d\mu(w) = \frac{(z \bullet \xi)^k}{N(k, n)}, \quad z \in \mathcal{R}, \xi \in \mathbb{C}^{n+1},
\]

where \( N(k, n) := \frac{(2k+n-1)(k+n-1)!}{k!(n-k)!} \).
3.1. The Bergman kernel

Next, let \( f \in \mathcal{O}(\mathfrak{R}) \) be a homogeneous polynomial of degree \( k \). Fix \( z \in \mathfrak{R} \). Then

\[
f(z) = C(k, n) \int_{\mathfrak{R}} \langle z, w \rangle^k f(w) \alpha(w) \wedge \overline{\alpha(w)},
\]

where

\[
C(k, n) := \frac{2(2k + n - 1)(n + k - 1)!}{\omega(\partial \mathfrak{R})(n - 1)! k!}.
\]

In fact, using (3.1.1) we have

\[
\int_{\mathfrak{R}} \langle z, w \rangle^k f(w) \alpha(w) \wedge \overline{\alpha(w)} = \omega(\partial \mathfrak{R}) \int_0^1 t^{2n-3+2k} dt \int_{\partial \mathfrak{R}} \langle z, w \rangle^k f(w) d\mu(w).
\]

Recall that \( f \) is a linear combination of a finite number of polynomials of the form \( z \longrightarrow (z \cdot \xi)^k \), \( \xi \in S^n \). So, what remains is to apply (3.1.2) to get the claim (3.1.3).

In order to be able to continue, we prove the following

**Lemma 3.1.10.** Let \( f \in \mathcal{O}(\mathfrak{R}) \). Then there are homogeneous polynomials \( f_k \) of degree \( k \), \( k \in \mathbb{N}_0 \), such that

\[
f(z) = \sum_{k=0}^{\infty} f_k(z), \quad z \in \mathfrak{R},
\]

and the convergence is uniform on compact subsets of \( \mathfrak{R} \).

**Proof.** Observe that \( A := \{0\} \cup \mathfrak{R} \) is an analytic subset of \( \mathbb{B} = \mathbb{B}_{n+1} \). It is clear that 0 is the only singularity of \( A \); it is a normal singularity for \( n \geq 2 \). Hence \( A \) is a normal complex space and the function \( f \) extends holomorphically to a function \( \tilde{f} \in \mathcal{O}(A) \).

Applying Cartan’s Theorem B we find an \( \tilde{f} \in \mathcal{O}(\mathbb{B}) \), \( \tilde{f}|A = f \). Therefore there are homogeneous polynomials \( f_k \) of degree \( k \) such that \( \tilde{f}(z) = \sum_{k=0}^{\infty} f_k(z), \ z \in \mathbb{B} \), and the convergence is locally uniform. \( \square \)

Denote by \( L^2(\mathfrak{R}) \) the space of all measurable functions on \( \mathfrak{R} \) satisfying

\[
\|f\|_{L^2(\mathfrak{R})} := \left( \int_{\mathfrak{R}} |f(z)|^2 \frac{\alpha(z) \wedge \overline{\alpha(z)}}{(1-\langle z, w \rangle)^{n+1}} \frac{(1-\langle z, w \rangle)^{n+1}}{(1+\langle z, w \rangle)^{n+1}} \right)^{1/2} < \infty,
\]

and let \( L^2_0(\mathfrak{R}) := L^2(\mathfrak{R}) \cap \mathcal{O}(\mathfrak{R}) \). Then we have the following formula for the Bergman kernel function of the space \( L^2_0(\mathfrak{R}) \).

**Lemma 3.1.11.** The Bergman kernel function is given by

\[
K_{\mathfrak{R}}(z, w) = \frac{2(-1)^n (n+1)}{\omega(\partial \mathfrak{R})(2i)^n} \left( \frac{(n-1)}{(1-\langle z, w \rangle)^{n+1}} + \frac{2n(\langle z, w \rangle)}{(1-\langle z, w \rangle)^{n+1}} \right), \quad z, w \in \mathfrak{R}.
\]

**Proof.** Fix \( z \in \mathfrak{R} \). Then, applying Lemma 3.1.10 and (3.1.3), we obtain

\[
f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} C(k, n) \int_{\mathfrak{R}} \langle z, w \rangle^k f_k(w) \alpha(w) \wedge \overline{\alpha(w)}
\]

\[
= \int_{\mathfrak{R}} K_{\mathfrak{R}}(z, w) f(w) \frac{\alpha(w) \wedge \overline{\alpha(w)}}{(1-\langle z, w \rangle)^{n+1}} \frac{(1+\langle z, w \rangle)^{n+1}}{(1-\langle z, w \rangle)^{n+1}}. \quad (3.1.4)
\]

Exploiting the last formula leads to the statement in Lemma 3.1.11. \( \square \)
To summarize, we have finished the first step of the proof of Theorem 3.1.9.

Now we continue with the second one.

Let \( \pi : \mathcal{R} \to \mathcal{M} \setminus \{0\} \), \( \pi(z, z_{n+1}) := (\tilde{z}, z) \in \mathcal{R} \). Then \( \pi \) is a proper map of degree 2. Its branching locus is called \( W; V := \pi(W) \). Let us denote the local inverses of \( \pi \) by \( \varphi \) and \( \psi \). They are given for \( z \in \mathcal{M} \setminus V \) by

\[
\varphi(z) = (z, i\sqrt{-1}z), \quad \psi(z) = (z, -i\sqrt{-1}z).
\]

Then a calculation leads to the following description of the pull back of \( \alpha \) under \( \varphi \) and \( \psi \) on \( \mathcal{M} \setminus V \):

\[
\varphi^{*}(\alpha) = \frac{n+1}{i\sqrt{-1}}(-1)^{n}dz_{1} \wedge \cdots \wedge dz_{n}, \quad \psi^{*}(\alpha) = \frac{n+1}{-i\sqrt{-1}}(-1)^{n}dz_{1} \wedge \cdots \wedge dz_{n}.
\]  

(3.1.5)

Let \( P_{\mathcal{R}} \) denote the Bergman projection on \( \mathcal{R} \) and \( P_{\mathcal{M}} \) the Bergman projection on \( \mathcal{M} \). Then we have the following relation.

**Lemma 3.1.12.** Let \( f \in L^{2}(\mathcal{M}) \). Then

\[
P_{\mathcal{R}}(\chi \cdot f \circ \pi)(z) = z_{n+1}P_{\mathcal{M}}(f)(\pi(z)), \quad z \in \mathcal{R},
\]

where \( \chi(z) := z_{n+1}, \ z \in \mathcal{R} \).

Now applying Lemma 3.1.12, we find a way to express the Bergman kernel function of \( \mathcal{M} \) in terms of the Bergman kernel function for \( \mathcal{R} \).

**Lemma 3.1.13.** Let \( \varphi \) and \( \psi \) be the local inverses from above. Then:

\[
z_{n+1}K_{\mathcal{M}}(\pi(z), w) = (n+1)^{2}\left(\frac{K_{\mathcal{R}}(z, \varphi(w))}{\varphi_{n+1}(w)} + \frac{K_{\mathcal{R}}(z, \psi(w))}{\psi_{n+1}(w)}\right), \quad z \in \mathcal{R}, \ w \in \mathcal{M} \setminus V.
\]

Proof. Fix a \( w \in \mathcal{M} \setminus V \) and choose a \( r > 0 \) such that \( w + rE^{n} \in \mathcal{M} \setminus V \). In view of Remark 6.1.4 in [J-P 1993], we may find a \( C^\infty \)-function \( u : \mathbb{C}^{n} \to [0, \infty), \) \( \supp u \subset w + rE^{n}, \) such that

\[
f(w) = \int_{\mathcal{M}} f(z)u(z)dA_{2n}(z), \quad f \in \mathcal{O}(\mathcal{M}).
\]

Therefore,

\[
K_{\mathcal{M}}(\cdot, w) = P_{\mathcal{M}}(u).
\]

Applying Lemma 3.1.12, it follows that

\[
z_{n+1}K_{\mathcal{M}}(\pi(z), w) = z_{n+1}P_{\mathcal{M}}(u)(\pi(z)) = P_{\mathcal{R}}(\chi \cdot u \circ \pi)(z)
\]

\[
= \int_{\mathcal{R}} \zeta_{n+1}u \circ \pi(\zeta)K_{\mathcal{R}}(z, \zeta)\frac{\alpha(\zeta) \wedge \alpha(\zeta)}{-(1)^{(n+1)}(2i)^{n}}
\]

\[
= (n+1)^{2}\int_{\mathcal{M} \setminus V} u(\eta)\left(\frac{K_{\mathcal{R}}(z, \varphi(\eta))}{\varphi_{n+1}(\eta)} + \frac{K_{\mathcal{R}}(z, \psi(\eta))}{\psi_{n+1}(\eta)}\right)dA_{2n}(\eta)
\]

\[
= (n+1)^{2}\left(\frac{K_{\mathcal{R}}(z, \varphi(w))}{\varphi_{n+1}(\eta)} + \frac{K_{\mathcal{R}}(z, \psi(w))}{\psi_{n+1}(\eta)}\right), \quad z \in \mathcal{R}.
\]

Hence the lemma is proved. \( \square \)
Now we are in the position to finish the proof of Theorem 3.1.9. According to Lemma 3.1.11, we have $K_R(z, w) = Ch(z \bullet w)$, where

$$C = \frac{2(2)^n(1 - \frac{1}{2})}{\omega(\partial G)} \quad \text{and} \quad h(t) := \frac{2n}{(1-t)^n} - \frac{n+1}{(1-t)^n}, \quad t \in \mathbb{C}.$$  

Hence,

$$K_R(\varphi(z), \varphi(w)) = Ch(x), \quad K_R(\varphi(z), \psi(w)) = Ch(y),$$

where $x := \langle z, w \rangle + t$, $t := \varphi_{n+1}(z)\varphi_{n+1}(w)$, and $y := \langle z, w \rangle - t$.

In virtue of Lemma 3.1.13, we get

$$K_M(z, w) = C(n+1)^2 \left( \frac{h(x) - h(y)}{t} \right).$$

Using the abbreviation $r := 1 - \langle z, w \rangle$, the last expression can be written as

$$Q := \frac{h(x) - h(y)}{t} = 2n \frac{(r+t)^{n+1} - (r-t)^{n+1}}{t(r^2-t^2)^{n+1}} - (n+1) \frac{(r+t)^n - (r-t)^n}{t(r^2-t^2)^n}.$$  

Then

$$Q = \frac{2n}{(r^2-t^2)^{n+1}} \left( \sum\limits_{k=0}^{n} \frac{n+1}{2k+1} r^{n-2k} t^{2k} - \frac{n+1}{(r^2-t^2)^n} \sum\limits_{k=0}^{\frac{n}{2}} \frac{n+1}{2k+1} r^{n-2k-1} t^{2k} \right).$$

Since $\binom{n}{2k+1} = \frac{n-2k}{2k+1} \binom{n+1}{2k+1}$, we proceed with our calculations and get

$$Q = \frac{2}{(r^2-t^2)^{n+1}} \sum\limits_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n+1}{2k+1} r^{n-1-2k} t^{2k} \left( 2nr - (n-2k)(r^2-t^2) \right),$$

which immediately leads to the formula in Theorem 3.1.9.  

**Example 3.1.14** (See Example 6.1.9 in [J-P 1993]). Applying Theorem 3.1.9, the biholomorphic mapping

$$M_2 \ni (z_1, z_2) \mapsto \frac{1}{\sqrt{2}}(z_1 + iz_2, z_1 - iz_2) \in G_2 := \{ z \in \mathbb{C}^2 : |z_1| + |z_2| < 1 \}$$

leads to the following formula of the Bergman kernel function of the domain $G_2$:

$$K_{G_2}(z, w) = \frac{2}{\pi^2} \cdot \frac{3(1 - \langle z, w \rangle)^2(1 + \langle z, w \rangle) + 4z_1 \bar{z}_2 \bar{w}_1 \bar{w}_2(5 - 3\langle z, w \rangle)}{((1 - \langle z, w \rangle)^2 - 4z_1 \bar{z}_2 \bar{w}_1 \bar{w}_2)^{\beta}}, \quad z, w \in G_2.$$  

Observe that this formula may be also derived from the one in Example 3.1.6(c) using the proper holomorphic mapping $D_2 \ni z \mapsto (z_1^2, z_2) \in G_2$ and Bell’s transformation law.

Fix points $z, w \in G_2$ and write, for abbreviation, $\xi_j := z_j \bar{w}_j$. Then $\sqrt{|\xi_1|} + \sqrt{|\xi_2|} < 1$, and so $4|\xi_1 \xi_2| < (1 - |\xi_1| - |\xi_2|)^2$. Therefore, the numerator in the formula above allows
the following estimate
\[ 3(1 - \langle z, w \rangle)^2(1 + \langle z, w \rangle) + 4z_1 \bar{z}_2 \bar{w}_1 \bar{w}_2(5 - 3\langle z, w \rangle) = 3(1 - \xi_1 - \xi_2)(1 - \xi_1 - \xi_2) + 8\xi_1\xi_2 \]
\[ \geq 3(1 - |\xi_1| - |\xi_2|)(1 - |\xi_1| - |\xi_2|)^2 - 2(1 - |\xi_1| - |\xi_2|)^2 \]
\[ \geq 3(1 - |\xi_1| - |\xi_2|)^2(1 + |\xi_1| - |\xi_2|) - 2(1 - |\xi_1| - |\xi_2|)^2 > (1 - |\xi_1| - |\xi_2|)^2 > 0. \]

Hence, the Bergman kernel function \( K_G \) has no zeros on \( G_2 \times G_2 \).

**Remark 3.1.15.** In [You 2002] an explicit formula for the Bergman kernel function is given even for a more general domain \( \Omega \) which could be thought as some interpolation between the minimal balls and the Euclidean balls. Here, we only describe \( \Omega \). Fix \( d \in \mathbb{N} \) and two \( d \)-tuples \( m = (m_1, \ldots, m_d) \in \mathbb{N}^d \) and \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \). Moreover, let \( a = (a_1, \ldots, a_d) \in [1, \infty)^d \). Then the domain \( \Omega = \Omega_{d,m,n,a} \) is given as

\[ \Omega := \left\{ Z = (Z(1), \ldots, Z(d)) \in M_{m_1,n_1}(\mathbb{C}) \times \cdots \times M_{m_d,n_d}(\mathbb{C}) : \sum_{j=1}^d \|Z(j)\|_{2a_j}^2 < 1 \right\}, \]

where \( M_{p,q}(\mathbb{C}) \) denotes the space of all \( p \times q \)-matrices with complex entities, and where

\[ ||M||_* := \left( \sum_{j=1}^p \left( \sum_{k=1}^q |z_{jk}|^2 + \sum_{k=1}^q |z_{jk}^2| \right) \right)^{1/2}, \quad M = (z_{jk})_{j=1, \ldots, p, k=1, \ldots, q} \in M_{p,q}(\mathbb{C}). \]

Observe that for \( d = 1 = a = m, n_1 = n \) the domain \( \Omega_{d,m,n,a} \) is just the minimal ball \( M \subset \mathbb{C}^n \).

### 3.2. The Lu Qi-Keng problem

For a while it was a question (posed by Lu Qi-Keng [LQK 1966]) whether the Bergman kernel function of a simply connected domain \( G \subset \mathbb{C}^n, n \geq 2 \), has no zeros. Such a domain is called a *Lu Qi-Keng domain*. A first example of a simply connected domain of holomorphy which is not a Lu Qi-Keng domain was given by H.P. Boas [Boa 1986] (see also [Skw 1980]). In fact, it turned out that the set of domains of holomorphy not being Lu Qi-Keng form a nowhere dense set in a suitable topology. For a more detailed discussion of this topic see [Boa 1990] (see also [Boa 2000]).

**Example 3.2.1.** Let \( D = D_p = \{ z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2/p} < 1 \} \), \( p \) a positive integer, be the third example of 3.1.6. Then there is the proper holomorphic mapping

\[ F : D_p \rightarrow G_p := \{ z \in \mathbb{C}^2 : |z_1| + |z_2|^{2/p} < 1 \}, \quad F(z_1, z_2) := (z_1^2, z_2). \]

Using Bell’s transformation law (see Theorem 3.1.4), we obtain

\[ K_{G_p}(z_1^2, 0), (w_1, 0))2z_1 = \left( K_D((z_1, 0), (\sqrt{w_1}, 0)) - K_D((z_1, 0), (-\sqrt{w_1}, 0)) \right) \frac{1}{2\sqrt{w_1}}, \]

whenever \( z_1 \in E, w_1 \in E \setminus \{0\} \).

Now, applying Example 3.1.6(c), it follows that

\[ K_{G_p}(z_1^2, 0), (w_1, 0))2z_1 = \frac{p + 1}{2w_1^{1/2}} \left( (1 - z_1w_1)^{-p/2} - (1 + z_1w_1)^{-p/2} \right). \]
Then, if $z_1 \neq 0$, the kernel function $K_{G_p}((z_1^2,0)(w_1^2,0))$ has a zero iff $(1 + x)^{p+2} = (1 - x)^{p+2}$, where $x := z_1 \overline{w_1}$. Observe that $\lambda \mapsto \frac{1+\lambda}{1-\lambda}$ maps $E$ biholomorphically to the right half-plane. Hence, $(\frac{1+\lambda}{1-\lambda})^{p+2} = 1$ has a non-zero solution iff $p > 2$.

We point out that also $K_{G_p}((0,0),(0,0))$ has zeros.

**Example 3.2.2.** Next, we study domains of the following type

$$\Omega_{n,m} := \left\{ (z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^n |z_j| + \sum_{k=1}^m |w_k|^2 < 1 \right\},$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

In a first step, let $n = 1$ and $m \in \mathbb{N}_0$. Then, using Bell’s transformation law for the proper holomorphic mapping $F : \mathbb{B}_k \rightarrow \Omega_{1,m}$, $F(z) := (z_1^2, z_2, \ldots, z_k)$, where $k := m+1$, we get

$$K_{\Omega_{1,m}}((z_1^2, z_2, \ldots, z_k), (w_2^2, w_3, \ldots, w_k)) = \frac{k!}{\pi^k 4z_1 \overline{w_1}} \left( \frac{1}{(1 - \langle z, w \rangle)^{k+1}} - \frac{1}{(1 + z_1 \overline{w_1} - \langle z, w \rangle)^{k+1}} \right),$$

where $\overline{z} := (z_2, \ldots, z_k)$ and $\overline{w} := (w_2, \ldots, w_k)$. In the case $m+2 > 4$, a similar reasoning as above gives $z_1, w_1 \in \mathbb{E}_s$ such that $K_{\Omega_{1,m}}((z_1^2, 0, \ldots, 0), (w_2^2, 0, \ldots, 0)) = 0$. If $m+2 \leq 4$, an easy calculation shows that $K_{\Omega_{1,m}}$ has no zeros on $\Omega_{1,m} \times \Omega_{1,m}$. Hence, the Bergman kernel function of $\Omega_{1,m}$ has a zero iff $m+2 > 4$.

Finally, using the above result for $n = 1$, induction over $n$, and the deflation method, we are led to the following result:

The Bergman kernel function of $\Omega_{n,m}$ has zeros iff $2n + m > 4$. In particular, the convex domain $\Omega_{n,0}$, $n \geq 3$, is not Lu Qi-Keng.

\[ \square \]

So far it is not known whether there is a convex domain in $\mathbb{C}^2$ which is not a Lu Qi-Keng domain.\[ \square \]

Let $n, k \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $\alpha \in (0,1]$. Put

$$N_{\alpha,k}(z) := \left( \sum_{\varepsilon_1, \ldots, \varepsilon_{n+1} \in \{+1,-1\}} a^{2\varepsilon_1 \ldots, \varepsilon_{n+1}}(z) + a^{2\varepsilon_1 \ldots, \varepsilon_{n+1}}(z) \right)^{\frac{1}{\alpha}}, \quad z \in \mathbb{C}^n \times \mathbb{C}^m,$$

where $\alpha_{\varepsilon_1, \ldots, \varepsilon_{n+1}}(z) := \sum_{j=1}^n \varepsilon_j |z_j| + \varepsilon_{n+1} \sum_{j=1}^m |z_{n+j}|^2$ and $\alpha(z) := \sum_{j=1}^{n+m} |z_j|^2$. Moreover, put

$$\Omega_{\alpha,k,n,m} := \{ z \in \mathbb{C}^{n+m} : N_{\alpha,k}(z) < 1 \}.$$

The following result is due to Nguyễn Viêt Anh [Vié 2000].

**Theorem 3.2.3.** The domain $\Omega_{\alpha,k,n,m}$ is strongly convex, algebraic \(^{(4)}\), complete Reinhardt. Moreover, if $2n - m > 4$, then there is a positive integer $M = M(a,n,m)$ such that for all $k \geq M$ the domain $\Omega_{\alpha,k,n,m}$ is not a Lu Qi-Keng domain.

In particular, for $m = 0$ there are strongly convex algebraic complete Reinhardt domains in $\mathbb{C}^n$, $n \geq 3$, which are not Lu Qi-Keng.

\(^{(3)}\) Recall Example 3.1.14.

\(^{(4)}\) Here "algebraic" means that the domain is given as the sublevel set of a real polynomial.
What are effective values for the number $M(a, n, m)$?

To prove Theorem 3.2.3, we need the following Lemma.

**Lemma 3.2.4.** Suppose $f_j : \mathbb{R}^q \rightarrow \mathbb{R}^+$ is a convex function, $j = 1, \ldots, p$. Then for $k \in \mathbb{N}$ the following function

$$
\rho(z) := \sum_{\varepsilon_1, \ldots, \varepsilon_p \in \{-1, 1\}} (\varepsilon_1 f_1(z) + \cdots + \varepsilon_p f_p(z))^{2k}, \quad z \in \mathbb{R}^q,
$$

is also a convex one.

**Proof.** Fix $z, w \in \mathbb{R}^q$. Then

$$
\frac{\rho(z) + \rho(w)}{2} \geq \sum_{\varepsilon_1, \ldots, \varepsilon_p \in \{-1, 1\}} \left( \varepsilon_1 f_1(z) + f_1(w) \right)^{2k} + \cdots + \left( \varepsilon_p f_p(z) + f_p(w) \right)^{2k}.
$$

Now recall the following formula

$$
\sum_{\varepsilon_1, \ldots, \varepsilon_p \in \{-1, 1\}} (\varepsilon_1 b_1 + \cdots + \varepsilon_p b_p)^{2k} = 2^p \sum_{k_1 + \cdots + k_p = k} \frac{(2k)!}{(2k_1)! \cdots (2k_p)!}.
$$

Plugging it into the first expression we get

$$
\frac{\rho(z) + \rho(w)}{2} \geq \sum_{k_1 + \cdots + k_p = k} \frac{(2k)!}{(2k_1)! \cdots (2k_p)!} \left( f_1(z) + f_1(w) \right)^{2k_1} + \cdots + \left( f_p(z) + f_p(w) \right)^{2k_p}.
$$

In virtue of the positivity and convexity of the functions $f_j$ the last inequality gives $(\rho(z) + \rho(w))/2 \geq \rho((z + w)/2)$, i.e. $\rho$ is a convex function.

**Proof of Theorem 3.2.3.** Put

$$
\rho(z) := \sum_{\varepsilon_1, \ldots, \varepsilon_{n+1} \in \{-1, 1\}} \alpha_{\varepsilon_1, \ldots, \varepsilon_{n+1}}^{2k}(z) + \alpha^{2k} - 1.
$$

Then $\rho$ is the defining function of the domain $\Omega = \Omega_{a,k,n,m}$. Using the above expansion, we see that $\rho$ is a polynomial with positive coefficients in $|z_1|^2, \ldots, |z_n|^2$ and $\sum_{j=1}^n |z_{n+j}|^2$. Hence, $\Omega$ is an algebraic complete Reinhardt domain with a smooth boundary. Moreover, in virtue of the Lemma 3.2.4, we may see that $\Omega$ is strongly convex.

Observe that $\Omega \subset \Omega_{n,m}$, where $\Omega_{n,m}$ is the domain from Example 3.2.2, and that $N_{a,k} \leq N_{a,l}$ when $l \leq k$. Moreover,

$$
\lim_{k \to \infty} N_{a,k}(z) = \sum_{j=1}^n |z_j| + \sum_{k=1}^m |z_{n+k}|^2, \quad z \in \Omega_{n,m}.
$$

What remains is to apply Ramadanov’s theorem (see [J-P 1993], Theorem 6.1.15), Example 3.2.2, and the Hurwitz theorem.

**Example 3.2.5.** We also mention that the minimal ball $\mathbb{M} \subset \mathbb{C}^n$, $n \geq 4$, is non Lu Qi-Keng [Pfl-You 1998]. This result is proved exploiting the explicit formula given in Theorem 3.1.9. In fact, let first $n \geq 5$: 

Put

\[ f : \mathbb{R} \to \mathbb{R}, \quad f(t) := -(n + 1) \arctan \frac{2t}{1 - t^2} + 2\pi - \arctan \frac{2(n^2 - 1)t}{(n - 1)^2 - (n + 1)^2t^2}. \]

Observe that \( f(0) = 2\pi \) and \( f(1/2) < 0 \) (here we need that \( n \geq 5 \)); so \( f(t_0) = 0 \) for a certain \( t_0 \in (0, 1/2) \). Therefore

\[ \left( \frac{1 - it_0}{1 + it_0} \right)^{n+1} = \frac{n - 1 + it_0(n + 1)}{n - 1 - it_0(n + 1)}. \]

Put \( z_0 := \sqrt{it_0}(1, 0, \ldots, 0), w_0 := \sqrt{-it_0}(0, 1, 0, \ldots, 0) \in \mathbb{C}^n \). A simple calculation gives that \( N(z_0) = N(w_0) = t_0 < 1/2 \); thus, \( z_0, w_0 \in \mathbb{M} \). Then, in virtue of Theorem 3.1.9, it follows that

\[ K_{\mathbb{M}}(z_0, w_0) = \frac{1}{n(n + 1)A_{2n}(\mathbb{M})} \frac{\sum_{j=0}^{n+1} (n+1)(n+2j)(n+2j+1)}{(1 - (it_0)^2)^{n+1}}. \]

Computing the binomial expression leads to

\[ K_{\mathbb{M}}(z_0, w_0) = \frac{(n - 1 + (n + 1)it_0)(1 + it_0)^{n+1} - (n - 1 - (n + 1)it_0)(1 - it_0)^{n+1}}{n(n + 1)A_{2n}(\mathbb{M})2it_0(1 - (it_0)^2)^{n+1}} = 0. \]

It remains the case \( n = 4 \): Consider the function

\[ g : \mathbb{R} \to \mathbb{R}, \quad g(s) := -28s^4 + 50s^3 - 10s^2 - 15s + 5. \]

Then \( g(0) = 5 \) and \( g(2/5) < 0 \). Therefore, there exists a \( s_0 \in (0, 2/5) \) with \( g(s_0) = 0 \). Put

\[ z_0 := \frac{\sqrt{s_0}(1 - 1)}{2}(i + \sqrt{i} - i + \sqrt{i}, 0, 0), \quad w_0 := \frac{\sqrt{s_0}(1 - i)}{2}(i - \sqrt{i}, -i - \sqrt{i}, 0, 0). \]

Then \( N(z_0) = N(w_0) < 1/2 \), i.e. \( z_0, w_0 \in \mathbb{M} \). A little calculation gives from the formula in Theorem 3.1.9 that

\[ K_{\mathbb{M}}(z_0, w_0) = \frac{g(s_0)}{5A_{2n}(\mathbb{M})((1 - s_0)^2 + s_0^2)^5} = 0. \]

Hence, the Bergman kernel function vanishes at the point \((z_0, w_0)\).

It is an open question whether the three dimensional minimal ball is a Lu Qi-Keng domain. For further open problems see also [Boa 2000]. Other examples of domains, that are not Lu Qi-Keng, may be found in [Die-Her 1999], [Eng 2000], and [Che 2002].

We close this section discussing consequences of the following result.

**Theorem 3.2.6** ([Eng 1997], [Eng 2000], [Che 2002]). Let \( D \subset \mathbb{C}^n \) be a bounded pseudoconvex domain, \( \varphi > 0 \) a positive function on \( D \), \( -\log \varphi \in PSH(D) \), such that \( 1/\varphi \in L_{loc}^1(D) \) fails to have a sesqui-holomorphic extension near a point \( z^0 \in D \) (i.e. there is no function \( f : V \times V \to \mathbb{C}, V \subset D \) a neighborhood of \( z^0 \), satisfying: \( f \) is holomorphic in the first coordinates and antiholomorphic in the latter, \( f(z, z) = 1/\varphi(z) \) for all \( z \in V \)).
Let $U = U(z^0) \subset D$ be a neighborhood. Then there is an $m_U \in \mathbb{N}$ such that the Bergman kernel function $K_{\Omega_m}((\cdot,0), (\cdot,0))$ of $\Omega_m := \{(z, w) \in D \times \mathbb{C}^m : \|w\|^2 < \varphi(z)\}$ has a zero in $U \times U$, $m \geq m_U$.

**Proof.** The proof of Theorem 3.2.6 is based on an extension theorem for $L^2$-functions (see [Ohs 2001]) and a description of the Bergman kernel with weights due to E. Ligocka (see [Lig 1989]).

Applying the first result we are led (3.5) to the following formula

$$
\lim_{k \to \infty} K_{\Phi^k}(z,z)^{1/k} = \frac{1}{\Phi(z)}, \quad z \in D,
$$

(3.2.7)

where $K_{\Phi^k}$ denotes the reproducing kernel function of the Hilbert space $L^2(D, \Phi^k)$. (Observe that in the case $\Phi \equiv 1$ this is just the classical Bergman kernel function.)

Then there is a $m_U$ such that $K_{\Phi^m}$ has zeros on $U \times U$, $m \geq m_U$. Otherwise, we may assume that $U$ is simply connected and that all the functions $K_{\Phi^k}$ have no zeros on $U \times U$. Next we choose a sesqui–holomorphic branch of $K_{\Phi^k}$ on $U \times U$. Since the function $1/\Phi$ is locally bounded, using (3.2.7) we see that the sequence $(K_{\Phi^k}(z,z)^{1/k})$ is locally bounded on $U$. Therefore, applying $|K_{\Phi^k}(z,w)|^2 \leq K_{\Phi^k}(z,z)K_{\Phi^k}(w,w)$, $z, w \in U$, shows that $(K_{\Phi^k})$ is locally bounded on $U \times U$. Hence, it converges locally uniformly to a sesqui–holomorphic function $L$ on $U \times U$ with $L(z,z) = 1/\Phi(z)$, $z \in U$; a contradiction.

It remains to recall Ligocka’s formula

$$
K_{\Omega_m}((z,t),(w,s)) = \sum_{j=0}^{\infty} \frac{(j+m)!}{j!m!} K_{\Phi^{m+j}}(z,w)(t,s)^j, \quad (z,t), (w,s) \in \Omega_m.
$$

Thus we have

$$
K_{\Omega_m}((z,0),(w,0)) = \frac{m!}{m} K_{\Phi^m}(z,w), \quad z, w \in D.
$$

Therefore, in virtue of the above claim, it follows that there is an $m_U$ such that for any $m \geq m_U$ the function $K_{\Omega_m}((\cdot,0), (\cdot,0))$ has zeros on $U \times U$. \qed

We should mention that the original formulation in [Che 2002] is much stronger as the one given here. Applying Theorem 3.2.6 for certain complex ellipsoids we obtain the following consequences.

**Corollary 3.2.7** ([Che 2002]). (a) For any $k \geq 1$, not an even integer, there exists an $m = m(k) \in \mathbb{N}$ such that

$$
\Omega := \Omega_k := \{(z,w) \in E \times \mathbb{C}^m : |z|^k + \|w\|^2 < 1\}
$$

is not Lu Qi-Keng.

(b) For any $k \in \mathbb{N}$ there exists a natural number $m = m(k)$ such that, if

$$
\Omega := \Omega_k := \{(z,w) \in E \times \mathbb{C}^m : |z|^{2k+1} + \|w\|^2 < 1\}, \quad (z^0, w^0) := (0,0,\ldots,-1) \in \partial \Omega,
$$

(3) We omit that proof.
then $\Omega$ is convex with a $C^k$-boundary and there are sequences

$$((z_j', w_j')), ((z_j'', w_j''))_j \subset \Omega, \quad \lim_{j \to \infty} (z_j', w_j') = \lim_{j \to \infty} (z_j'', w_j'') = (z^0, w^0),$$

such that $K_\Omega((z_j', w_j'), (z_j'', w_j'')) = 0$, $j \in \mathbb{N}$. In particular, the set $\{(z, w) \in \Omega \times \Omega : K_\Omega(z, w) = 0\}$ accumulates at $((z^0, w^0), (z^0, w^0))$.

**Proof.** (a) Take $D = E$ and $\varphi(z) := 1 - |z|^k$, $z \in E$. Then $-\log \varphi \in \mathcal{SH}(E)$ and $1/\varphi$ is not real analytic at 0. So it cannot be extended to a sesqui-holomorphic function near 0. Hence, in virtue of Theorem 3.2.6, there is a neighborhood $U = U(0)$ and an $m = m(k) \in \mathbb{N}$ such that $K_D((\cdot, 0), (\cdot, 0))$ has at least one zero in $U \times U$.

(b) Fix a $k$. In virtue of part (a), there is an $m = m(k)$ such that $K_\Omega$ has a zero at a point $((z', w'), (z'', w'')) \in \Omega \times \Omega$.

Put

$$D := \left\{ \zeta \in \mathbb{C}^{m+1} : |\zeta| \frac{2}{m+1} + \sum_{j=2}^m |\zeta_j|^2 + \Re \zeta_{m+1} < 0 \right\}.$$ 

Observe that

$$\Phi(\zeta) := \left( \frac{4\pi^2}{(\zeta_{m+1} - 1)^2}, \frac{2\zeta_2}{\zeta_{m+1} - 1}, \ldots, \frac{2\zeta_{m+1}-1}{\zeta_{m+1} - 1}, \frac{\zeta_{m+1}-1}{\zeta_{m+1}-1} \right)$$

defines a biholomorphic map from $D$ to $\Omega$.

Moreover, for any positive $\varepsilon$,

$$F_{\varepsilon}(\zeta) := (\varepsilon \frac{2}{m+1} \zeta_1, \sqrt{\varepsilon} \zeta_2, \ldots, \sqrt{\varepsilon} \zeta_m, \varepsilon \zeta_{m+1})$$

is a biholomorphic mapping from $D$ to $D$. Therefore,

$$K_D(\Phi \circ F_{\varepsilon} \circ \Phi^{-1}(z', w'), \Phi \circ F_{\varepsilon} \circ \Phi^{-1}(z'', w'')) = 0, \quad \varepsilon > 0.$$ 

It remains to mention that $\lim_{\varepsilon \to 0} \Phi \circ F_{\varepsilon} \circ \Phi^{-1}(z', w') = \lim_{\varepsilon \to 0} \Phi \circ F_{\varepsilon} \circ \Phi^{-1}(z'', w'') = (z^0, w^0)$.

[*] It would be interesting to find in the situation of Corollary 3.2.7 concrete numbers $m = m(k)$.

So far, we saw that some of the domains $\Omega_p \subset \mathbb{C}^n$ are not Lu Qi-Keng, some of them are. [*] Describe all the vectors $p = (p_1, \ldots, p_n)$ for which the Bergman kernel function of $\Omega_p$ is zero-free [*]

**Remark 3.2.8.** Recall that in the situation of Corollary 3.2.7 (b) we have

$$\lim_{z \to z^0} K_\Omega(z) = \lim_{z \to z^0} K_\Omega(z, z) = \infty$$

(apply Theorem 6.1.17 in [J-P 1993]). Therefore, Corollary 3.2.7 (b) shows that $K_\Omega$ does not continuously extend as a map $\overline{\Omega} \times \overline{\Omega} \to \mathbb{C}$. [*] It is unknown whether this negative phenomenon does also occur for $C^\infty$–smooth convex domains [*]

In addition to Remark 3.2.8 we recall that the Bergman kernel function $K_D$, $D \subset \mathbb{C}^n$ a smooth bounded strictly pseudoconvex domain, can be smoothly extended to $\overline{D} \times \overline{D} \setminus \nabla(\partial D)$, where $\nabla(\partial D) := \{(z, z) : z \in \partial D\}$ (see [Ker 1972]). This result was generalized by Bell and Boas (see [Bel 1986], [Boa 1987]) to the following statements:
(a) Let \( D \subset \mathbb{C}^n \) be a smoothly bounded pseudoconvex domain. Let \( \Gamma_1, \Gamma_2 \subset \partial D \) be two open disjoint subsets of the boundary consisting of points of finite type (in the sense of d’Angelo). Then \( K_D \) extends smoothly to \((D \cup \Gamma_1) \times (D \cup \Gamma_2)\).

(b) Let \( D \) be as in (a) and assume that \( D \) satisfies condition (R) (\(^{6}\)). If \( \Gamma_1, \Gamma_2 \) are disjoint open subsets of \( \partial D \) and \( \Gamma_1 \) consists of points of finite type, then \( K_D \) extends smoothly to \((D \cup \Gamma_1) \times (D \cup \Gamma_2)\).

There was the question whether a similar extension phenomenon might be probable for any smoothly bounded pseudoconvex domain. That this is not true is shown by So-Chin Chen [Chen 1996].

**Theorem 3.2.9.** Let \( D \subset \mathbb{C}^n \) be a smoothly bounded pseudoconvex domain, \( n \geq 2 \). Suppose that its boundary contains a non-trivial complex variety \( V \). Then \( K_D \) cannot continuously extended to \( \overline{D} \times \overline{D} \setminus \nabla(\partial D) \).

**Proof.** Take a regular point \( z^0 \in V \) and denote by \( n \) the outward unit normal at \( z^0 \). Then the smoothness assumption gives an \( \varepsilon_0 > 0 \) such that
\[
w - \varepsilon n \in D, \quad \varepsilon \in (0, \varepsilon_0), \quad w \in \partial D \cap B(z^0, \varepsilon_0).
\]

Moreover, we choose a holomorphic disc in \( V \), i.e. a holomorphic embedding \( \varphi : E \longrightarrow V \), with \( \varphi(0) = z^0 \) and \( \varphi(E) \subset V \cap B(z^0, \varepsilon_0) \).

Now assume that \( K_D \in C(\overline{D} \times \overline{D} \setminus \nabla(\partial D)) \). Then
\[
\sup_{|\lambda|=1/2} |K_D(z^0, \varphi(\lambda))| < \infty.
\]

Applying Theorem 6.1.17 in [J-P 1993] and the maximum principle leads to
\[
\sup_{|\lambda|=1/2} |K_D(z^0, \varphi(\lambda))| = \lim_{\varepsilon \to 0} \sup_{|\lambda|=1/2} |K_D(z^0 - \varepsilon n, \varphi(\lambda) - \varepsilon n)| \\
\geq \lim_{\varepsilon \to 0} K_D(z^0 - \varepsilon n, z^0 - \varepsilon n) = \infty;
\]
a contradiction. \( \square \)

**Example 3.2.10** ([Chen 1996]). Fix a smooth real valued function \( r : \mathbb{R} \longrightarrow \mathbb{R} \) with the following properties:

(i) \( r(t) = 0 \) if \( t < 0 \),

(ii) \( r(t) > 1 \) if \( t > 1 \),

(iii) \( r''(t) \geq 100r'(t) \) for all \( t \),

(iv) \( r''(t) > 0 \) if \( t > 0 \),

(v) \( r'(t) > 100 \), if \( r(t) > 1/2 \).

For \( s > 1 \) put
\[
\Omega := \Omega_s := \{ z \in \mathbb{C}^2 : \rho(z) < 0 \}, \quad \text{where} \quad \rho(z) := \rho_s(z) := |z_1|^2 - 1 + r(|z_2|^2 - s^2).
\]

Then \( \Omega_s \) is a smoothly bounded pseudoconvex domain in \( \mathbb{C}^2 \), it is convex and satisfies condition (R), and it is strictly bounded pseudoconvex everywhere except on the set
\[
\{ z \in \mathbb{C}^2 : |z_1| = 1, \ 0 \leq |z_2| \leq s \} \subset \partial \Omega.
\]

\(^{6}\) A bounded domain is said to satisfy condition (R) if the Bergman projection \( L^2(D) \longrightarrow L^2_{\mathbb{R}}(D) \) sends \( C^\infty(\overline{D}) \cap L^2(D) \to C^\infty(\overline{D}) \cap \mathcal{O}(D) \).
3.3. Bergman exhaustiveness

Obviously, this set contains non-trivial analytic varieties. So \( \Omega \) is an example for a domain treated in Theorem 3.2.9.

### 3.3. Bergman exhaustiveness

In the study of the Bergman kernel it is important to know its boundary behavior. We define

**Definition 3.3.1.** Let \( D \subset \mathbb{C}^n \) be a domain and \( z^0 \in \partial D \). We say that \( D \) is **Bergman exhaustive at** \( z^0 \) (for short, \( b \)-exhaustive) if \( \lim_{D \ni z \to z^0} k_D(z) = \infty \). Moreover, if \( D \) is \( b \)-exhaustive at any of its boundary points, then \( D \) is called **\( b \)-exhaustive**.

Obviously, any \( b \)-exhaustive domain is pseudoconvex. There are a lot of general results giving sufficient condition for a pseudoconvex domain to be \( b \)-exhaustive at a boundary point. Besides Theorem 6.1.17 in [J-P 1993] the most general is the following one that relates \( b \)-exhaustiveness to the boundary behavior of certain level sets of the Green function. For an arbitrary domain \( D \subset \mathbb{C}^n \) and a point \( z \in D \) we define

\[
A_z := A_z(D) := \{ w \in D : \log g_D(z, w) \leq -1 \}.
\]

Then:

**Theorem 3.3.2.** Let \( D \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \) and \( z_0 \in \partial D \). Assume that
\[
\lim_{z \to z_0} \Lambda \frac{2}{n} (A_z(D)) = 0.
\]

Then \( D \) is \( b \)-exhaustive at \( z_0 \).

Theorem 3.3.2 is a simple consequence of the following result ([Che 1999], [Her 1999]).

**Theorem 3.3.3.** For any \( n \in \mathbb{N} \) there exists a positive number \( C_n \) such that for every bounded pseudoconvex domain \( D \subset \mathbb{C}^n \) the following is true:

\[
\frac{|f(z)|^2}{k_D(z)} \leq C_n \int_{A_z} |f(w)|^2 dA_{2n}(w), \quad f \in L^2(D), \quad z \in D.
\]

**Proof.** Let \( D \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \), \( z_0 \in D \), and fix an \( f \in L^2(D) \), \( f \neq 0 \). Put

\[ D_t := \{ z \in D : \text{dist}(z, \partial D) > t \}, \quad 0 < t < 1 \text{ sufficiently small}. \]

Moreover, let \( \psi_1 \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{R}) \) be a non negative polyradial symmetric function with

\[ \int_{\mathbb{C}} \psi(z) dA_{2n}(z) = 1 \text{ and supp } \psi_1 \subset \mathbb{B}_n(0, 1); \text{ put } \psi_t(z) := \frac{1}{t^n} \psi_1(\frac{z}{t}), \quad z \in \mathbb{C}^n, \quad t > 0. \]

On \( D_t \) we define

\[ \varphi_t(z) := 2n V_t(z) + \exp(V_t(z)) + t \|z\|^2, \quad \varphi(z) := 2n \log g_D(z_0, \cdot) + g_D(z_0, \cdot), \]

where \( V_t := \log g_D(z_0, \cdot) * \psi_t \). Finally, we choose a \( \chi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1]) \) with \( \chi(t) = 1 \) if \( t \leq -2 \), \( \chi(t) = 0 \) if \( t \geq -1 \), and \( |\chi'| \leq 2 \).

We define the following \( \overline{\partial} \)-closed \( (0, 1) \)-form \( \alpha_t \) on \( D_t \),

\[ \alpha_t := \overline{\partial}(\chi \circ V_t \cdot f) = \chi'(V_t)f\overline{\partial}V_t. \]
Observe that \( \alpha_t \) is a smooth form whose support is contained in the set \( \{ -2 \leq V_t \leq -1 \} \). Moreover, \( \varphi_t \geq -4n \) on \( \supp \alpha_t \). Therefore, \( \int_{D_t} |\alpha_t|^2 e^{-\varphi_t} dA_{2n} < \infty \).

For the Levi form of \( \varphi_t \) we have the following estimate
\[
L \varphi_t(z; X) \geq e^{V_t(z)}|V_t'(z)|X^2 \geq e^{-2}|V_t'(z)|X^2, \quad z \in \supp \alpha_t, \ X \in \mathbb{C}^n.
\]

Let \( Q \) denote the inverse matrix of the coefficient matrix of \( L \). Then, if \( z \in \supp \alpha_t \), we have
\[
\sum_{j,k=1}^n Q_{jk}(z)\alpha_{ij}(z)\alpha_{ik}(z) \exp(-\varphi_t(z)) \leq e^2|\chi'(V_t(z))|^2|f(z)|^2 e^{-\varphi_t(z)} \leq 4e^{4n+2}|f(z)|^2.
\]

Therefore, in virtue of Lemma 4.4.1 in [Hör 1979], there exists a solution \( u_t \in \mathcal{C}^\infty(D_t) \) of the equation \( \tilde{\vartheta} u_t = \alpha_t \) with the following estimates
\[
\int_{D_t} |u_t|^2 e^{-\varphi_t} dA_{2n} \leq 4e^{4n+2} \int_{\supp \alpha_t} |f|^2 dA_{2n}.
\]

Put
\[
v_t := \begin{cases} u_t e^{-\varphi_t/2} & \text{on } D_t \\ 0 & \text{on } D \setminus D_t. \end{cases}
\]

Then the family \( (v_t)_t \) belongs to \( L^2(D) \) and satisfies the following uniform estimate
\[
\int_D |v_t|^2 dA_{2n} \leq 4e^{4n+2} \int_{A_{t_0}} |f|^2 dA_{2n}
\]
(observe that \( \supp \alpha_t \subset \{ -2 \leq V_t \leq -1 \} \subset A_{t_0} \)).

In virtue of the Alaoglu–Bourbaki theorem, we may find a function \( v \in L^2(D) \) satisfying
\[
\int_D |v|^2 dA_{2n} \leq 4e^{4n+2} \int_{A_{t_0}} |f|^2 dA_{2n}.
\]

Put \( u := v e^{\varphi/2} \). Then
\[
\int_D |u|^2 dA_{2n} \leq e \int |v|^2 dA_{2n} \leq 4e^{4n+3} \int_{A_{t_0}} |f|^2 dA_{2n}. \tag{3.3.8}
\]

Using distributional derivatives, we find an \( \tilde{f} \in \mathcal{O}(D) \) such that
\[
\tilde{f} = \chi \circ \log g_D(z_0, \cdot) - u
\]
almost everywhere on \( D \). Moreover, take a neighborhood \( U \subset D \) of \( z_0 \), \( \log g_{D(z_0, \cdot)} \cdot f \leq -3 \) on \( U \). Then \( f - \tilde{f} = u \) almost everywhere on \( U \). In virtue of (3.3.8), it follows that
\[
\int_U |f - \tilde{f}|^2 e^{-\varphi} dA_{2n} < \infty.
\]

Observe that \( e^\varphi \) is not locally integrable near \( z_0 \); hence \( \tilde{f}(z_0) = f(z_0) \).

Summarizing, we have found an \( \tilde{f} \in L^2(D) \) with \( f(z_0) = \tilde{f}(z_0) \) and
\[
\|\tilde{f}\|_{L^2(D)} \leq (1 + 4e^{4n+3}) \int_{A_{t_0}} |f|^2 dA_{2n}.
\]
3.3. Bergman exhaustiveness

Consequently,
\[
\frac{|f(z_0)|^2}{k_D(z_0)} \leq \|f\|^2_{L^2(D)} \leq (1 + 4e^{4n+3}) \int_{A_{\infty}} |f|^2 dA_{2n},
\]
which finishes the proof. \(\Box\)

**Proof of Theorem 3.3.2.** In virtue of Theorem 3.3.3 we know that there is a constant \(C_n > 0\) such that
\[
\frac{1}{k_D(z)} \leq C_n \int_{A_z} dA_{2n}(w) \leq C_n A_{2n}(A_z(D)) \longrightarrow 0;
\]
Therefore, \(k_D(z) \longrightarrow \infty.\) \(\Box\)

Moreover, combining Theorem 3.3.2 and a result due to B/suppress locki we have the following (see also [Ohs 1993]).

**Theorem 3.3.4.** For a bounded hyperconvex domain \(D \subset \mathbb{C}^n\) (i.e. there is a negative \(u \in PSH(D)\) such that the sublevel sets \(\{z \in D : u(z) < -\varepsilon\}, \varepsilon > 0,\) are relatively compact in \(D\)), the following is true: \(A_{2n}(A_z(D)) \longrightarrow 0.\) In particular, any hyperconvex domain is \(b-\)exhaustive.

**Proof.** According to [Blo 1996], there is a function \(u \in C(\overline{D}) \cap PSH(D)\) satisfying the following properties:
\[
u|_{\partial D} = 0 \text{ and } (dd^c u)^n \geq A_{2n}.
\]
Applying [Blo 1993], we get for a point \(z_0 \in \partial D:\)
\[
\int_D (-log g_D(z, w))^n dA_{2n}(w) \leq \lim_{k \to \infty} \int_D (-\max\{log g_D(z, \cdot), -k\})^n (dd^c u)^n
\leq n!\|u\|^{n-1}_{L^\infty(D)} |u(z)| \longrightarrow 0,
\]
where the last inequality is due to Demailly (see [Dem 1987]).

Finally, in virtue of Theorem 3.3.3, we get
\[
\frac{1}{k_D(z)} \leq C_n \int_{A_z(D)} dA_{2n}(w) \leq C_n \int_D (-log g_D(z, w))^n dA_{2n}(w) \longrightarrow 0.
\]
Since \(z_0\) is arbitrary, it follows that \(k_D(z) \longrightarrow \infty,\) i.e. \(D\) is \(b-\)exhaustive. \(\Box\)

**Example 3.3.5.** (1) There is a large class of bounded pseudoconvex domains which are hyperconvex, namely

**Theorem 3.3.6 ([Ker-Ros 1981], [Dem 1987]).** Any bounded pseudoconvex domain \(D \subset \mathbb{C}^n\) with a Lipschitz boundary is hyperconvex. In particular, if \(D\) has a \(C^1-\)boundary, then it is hyperconvex.

(2) Hyperconvexity is even a local property.

**Theorem 3.3.7 ([Ker-Ros 1981]).** Suppose that \(D\) is a bounded domain in \(\mathbb{C}^n\) such that every \(z_0 \in \partial D\) has a neighborhood \(U = U(z_0)\) for which \(D \cap U\) is hyperconvex. Then \(D\) itself is hyperconvex.
(3) Put $D := \{ z \in \mathbb{C}^2 : |z_1| < |z_2| < 1 \}$. Then $D$ is $b$–exhaustive but not hyperconvex. (For other examples of this type see also Theorems 3.3.8 and 3.3.9 and Example 3.3.23.) For $D$ even more is true. Namely, there is a sequence $(z_k)_k \subset D$ tending to $0$ such that $A_{2n}(A_z(D))_k \not\to 0$.

For Reinhardt domains in $\mathbb{C}^2$ we have (see [Zwo 2001a]) the following general result for the pole boundary behavior of the Green function.

**Theorem 3.3.8.** Let $D \subset \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain such that $D \cap (\mathbb{C}^* \times \{0\}) = E_+ \times \{0\}$. Moreover, suppose that for a $z^0 \in D$:

$$\forall v \in \mathbb{R}^2 : (\log |z_1^0|, \log |z_2^0|) + \mathbb{R}_+ v \subset \log D = \mathbb{R}_+(0,-1).$$

Then

$$g_D(z, w) \quad \underset{D \ni z \to 0}{\longrightarrow} \quad 0, \quad w \in D \cap \mathbb{C}_+, \quad \text{and, therefore, } A_{2n}(A_z(D)) \underset{D \ni z \to 0}{\longrightarrow} 0.$$

In particular, $D$ is $b$–exhaustive at the origin but not hyperconvex.

**Proof.** We may assume that $D = \{ z \in E^2 : |z_2| < \rho(|z_1|) \}$, where $\rho : [0,1) \to [0,1]$, $\rho(r) = 0$ iff $r = 0$ and

$$\forall A > 0 \exists B \in \mathbb{R} : \log \rho(e^t) \leq At + B, \quad t \in -\mathbb{R}_+.$$

Take a $z \in D$ close to $0$. Then for $w \in D$ with $|w_1| = 2|z_1|$ we have

$$g_D(z, w) \geq g_E(z_1, w_1) \geq \frac{|z_1 - w_1|}{|1 - z_1 w_1 |} \geq \frac{|z_1|}{2}.$$ 

Now we claim that

$$\log g_D(z, w) \geq \log |z_1| - \frac{\log |w_2|}{2 \log \rho(2|z_1|)}, \quad w \in D, \quad |w_1| \geq 2|z_1|. \quad (3.3.9)$$

If $|w_1| = 2|z_1|$ then (3.3.9) is true since the second factor is larger or equal $1$. Moreover, in virtue of Remark 2.5.6, we know that $c_D(z, w) \to \infty$ whenever $w \to w^*, w^* \in \partial D \cap \mathbb{C}^2$.

In particular, $g_D(z, w) \to 1$. Therefore, the maximality of the Green function implies inequality (3.3.9).

It remains to show that the right side of (3.3.9) tends to $0$ if $z$ tends to $0$. In fact, the right side can be written with $t = \log |z_1| < 0$ and $v(t) := \log \rho(e^t)$ as

$$\frac{t - \log 2}{v(t + \log 2)} \log |w_2| = \frac{t - \log 2}{t + \log 2} \log |w_2| - \frac{t + \log 2}{v(t + \log 2)} =: f(t).$$

According to our assumption, we know that for any $A > 0$ we have $\lim_{t \to -\infty} f(t) \geq A$. In particular, since $A$ is arbitrary, $\lim_{t \to -\infty} \frac{v(t)}{t} = \infty$. Therefore, $\lim_{t \to -\infty} f(t) = 0$, which finishes the proof.

A Reinhardt domain satisfying the conditions of Theorem 3.3.8 is given, for example, by $D := \{ z \in E_+ \times E : |z_2| < e^{-1/|z_1|} \}$. Hence, $D$ is $b$–exhaustive but not hyperconvex.

For circular domains we have the following result.
Theorem 3.3.9 ([Jar-Pfl-Zwo 2000]). Any bounded pseudoconvex balanced domain is $b$-exhaustive.

Proof. Let $D = D_h = \{ z \in \mathbb{C}^n : h(z) < 1 \}$ be a bounded pseudoconvex balanced domain. Fix a boundary point $z_0$ and let $M$ be an arbitrary positive number. Put $H := C_{z_0}$. Then, in virtue of the theorem of Ohsawa (see Theorem 3.1.1), we have $k_{D \cap H}(z) \leq C k_D(z)$, $z \in D \cap H$, where $C$ is a suitable positive number. Since $D \cap H$ is a plane disc, there is an $s \in (0,1)$ such that $M < k_{D \cap H}(sz_0)$. Using the continuity of $k_D$ leads to an open neighborhood $U = U(z_0) \subset D \setminus \{0\}$ such that $k_D(z) > M$, $z \in U$.

Now fix a $z \in U$ and define $u_z : \frac{1}{h(z)}E \rightarrow \mathbb{R}$, $u_z(\lambda) := k_D(\lambda z)$. This function is subharmonic and radial, so $u|_{0,\frac{1}{h(z)}}$ is an increasing function. Therefore, $M < u_z(1) \leq u_z(\lambda) = k_D(\lambda z)$, $1 \leq |\lambda| < \frac{1}{h(z)}$. Obviously, 

$$V = V_{z_0,M} := \{ \lambda z : z \in U, \lambda \in \mathbb{C}, |\lambda| > 1 \}$$

is an open neighborhood of $z_0$. Since $M$ is arbitrary, we have $\liminf_{D \ni z \rightarrow z_0} k_D(z) = \infty$ proving the theorem. \qed

In the case of a bounded pseudoconvex balanced domain with a continuous Minkowski function, Theorem 3.3.9 was proved in [Jar-Pfl 1989] (see Theorem 7.6.7 in [J-P 1993]).

Observe that any bounded hyperconvex balanced domain is taut and therefore its Minkowski function $h$ is continuous. Obviously, there are a lot of bounded balanced pseudoconvex domains with a non-continuous Minkowski function. Moreover, we mention that there exists a bounded pseudoconvex balanced domain $D$ which is not fat (i.e. $\text{inf} D \neq D$); see Example 3.1.12 in [J-P 1993].

Describe all bounded pseudoconvex circular domain $D$ (i.e. $\forall z \in D, \theta \in \mathbb{R} : e^{i\theta}z \in D$) which are $b$-exhaustive.

Example 3.3.10. Let $D \subset \mathbb{C}^n$ be a bounded domain, $H : D \times \mathbb{C}^m \rightarrow [0,\infty)$ such that $\log H \in \mathcal{PSH}(D \times \mathbb{C}^m)$, $H(z,\lambda w) = |\lambda|H(z,w)$, $(z,w) \in D \times \mathbb{C}^m$ and $\lambda \in \mathbb{C}$. Put $G_D := \{ (z,w) \in D \times \mathbb{C}^m : H(z,w) < 1 \}$. $G_D$ is a Hartogs domain with $m$-dimensional fibers. Assume that $G_D$ is bounded and pseudoconvex. Then we have the following result.

Theorem 3.3.11. Let $G_D$ be bounded pseudoconvex as above and let $(z_0, w_0) \in \partial G_D$. Assume that one of the following conditions is satisfied:

(a) $z_0 \in D$,
(b) $z_0 \in \partial D$ and $\lim_{D \ni z \rightarrow z_0} k_D(z) = \infty$,
(c) there is a neighborhood $U = U((z_0, w_0))$ such that $U \cap G_D \subset \{ (z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\| < \|z - z_0\|^{\delta} \}$

for some $\delta > 0$.

Then $\lim_{G_D \ni (z,w) \rightarrow (z_0, w_0)} k_{G_D}(z,w) = \infty$. In particular, if $D$ is $b$-exhaustive, then so is $G_D$.

For a proof see [Jar-Pfl-Zwo 2000].
Example 3.3.12. The following example shows that Theorem 3.3.11 is far away from being optimal. Fix sequences \((a_j)_{j \in \mathbb{N}} \subset (0, 1)\) and \((n_j)_{j \in \mathbb{N}} \subset \mathbb{N}\) with \(\lim_{j \to \infty} a_j = 0\) and \(n_j \geq j\). Put on \(E_k := E \setminus \{a_j : j = 1, \ldots, k\}\), \(u_k(\lambda) := \sum_{j=1}^{k} \left(\frac{a_j}{2^j - a_j}\right)^{n_j}\). Observe that \(u_k(0) < 0\). Define \(E_\infty := E \setminus (\{0\} \cup \{a_j : j \in \mathbb{N}\})\). Then the sequence \((u_k)_k\) is locally bounded from above on \(E_\infty\) and globally bounded from below; moreover, it is an increasing sequence of subharmonic functions. It turns out that \(u := \lim_{k \to \infty} u_k \in \mathcal{SH}(E_\infty)\) and \(\lim_{\{(-1,0)\}} u(x) \leq 0\). Finally, we define the following bounded pseudoconvex Hartogs domain with one-dimensional fibers

\[G_{E_\infty} := \{(z, w) \in E_\infty \times \mathbb{C} : |w| < e^{-u(z)}\}.
\]

Obviously, the point \((0,0) \in \partial G_{E_\infty}\) does not satisfy any of the conditions in Theorem 3.3.11. Nevertheless, a correct choice of the \(n_j\)'s may show that \(G_{E_\infty}\) satisfies the cone condition of Theorem in [J-P 1993] at \((0, 0)\). Therefore,

\[k_{G_{E_\infty}}((z, w)) \underset{G_{E_\infty} \not
subseteq \{z,w\} \to (0,0)}{\longrightarrow} \infty.
\]

The discussion of the other boundary points with the help of Theorem 3.3.11 and Theorem 6.1.17 in [J-P 1993] even proves that \(G_{E_\infty}\) is \(b\)-exhaustive. Try to give a complete description of those bounded pseudoconvex Hartogs domains with \(m\)-dimensional fibers that are \(b\)-exhaustive.

In the complex plane there is even a full characterization of bounded domains being \(b\)-exhaustive in terms of the potential theory (see [Zwo 2002]). To be able to present this result we recall a few facts from the classical plane potential theory.

3.3.1. A short course in plane potential theory. (See [Ran 1995]) Let \(K \subset \mathbb{C}\) be compact and \(\mathcal{P}(K) := \{\mu : \mu \text{ a probabilistic measure of } K\}\). For \(\mu \in \mathcal{P}(K)\),

\[p_\mu(\lambda) := \int_K \log |\lambda - \zeta| d\mu(\zeta), \quad \lambda \in \mathbb{C},\]

is the logarithmic potential of \(\mu\). Recall that \(p_\mu \in \mathcal{SH}(\mathbb{C})\) and that \(p_\mu|_{\mathbb{C} \setminus K}\) is a harmonic function. To any such a \(\mu\) one associates its energy

\[I(\mu) := \int_K p_\mu(\lambda)d\mu(\lambda) = \int_K \int_K \log |\lambda - \zeta| d\mu(\lambda)d\mu(\zeta).
\]

A probabilistic Borel measure \(\nu \in \mathcal{P}(K)\) is called the equilibrium measure of \(K\) if \(I(\nu) = \sup_{\mu \in \mathcal{P}(K)} I(\mu)\). It is known that the equilibrium measure exists and is unique if \(K\) is not a polar set; then we write \(\nu_K\). Moreover, the logarithmical capacity of any set \(M \subset \mathbb{C}\) is given by

\[\text{cap}(M) := \exp(\sup\{I(\mu) : K \subset M \text{ compact }, \mu \in \mathcal{P}(K)\}).\]

In the case when \(M = K\) compact and not polar then \(\text{cap}(K) = e^{I(\nu_K)}\). Moreover, if \(M\) is any Borel set then: \(M\) is polar iff \(\text{cap}(M) = 0\).

For further applications we collect a few well known properties of the logarithmic capacity:

1. if \(M_1 \subset M_2\) then \(\text{cap}(M_1) \leq \text{cap}(M_2)\);
2. if \( M_1 \subset M_2 \subset M_3 \subset \ldots \) are Borel sets then \( \text{cap}(\bigcup_{j=1}^{\infty} M_j) = \lim_{j \to \infty} \text{cap}(M_j) \);  
3. if \( K_1 \supset K_2 \supset K_3 \ldots \) are compact sets, then \( \text{cap} K_k \to \cap(\bigcap_{k=1}^{\infty} K_k) \);  
4. if \( M = \bigcup_{j=1}^{N} M_j, M_j \) Borel sets with \( \text{diam} M \leq d, N \in \mathbb{N} \cup \{ \infty \} \), then  
\[
\frac{1}{\log d - \log \text{cap} M} \leq \sum_{j=1}^{N} \frac{1}{\log d - \log \text{cap} M_j};
\]

(4') if \( M = \bigcup_{j=1}^{N} M_j, M_j \) Borel sets with \( \text{dist}(M_j, M_k) \geq d > 0, k \neq j, N \in \mathbb{N} \cup \{ \infty \} \), then  
\[
\frac{1}{\log^+ \left( \frac{d}{\text{cap} M} \right)} \geq \sum_{j=1}^{N} \frac{1}{\log^+ \left( \frac{d}{\text{cap} M_j} \right)};
\]

5. **Theorem of Frostman.** Let \( K \subset \mathbb{C} \) be a non polar compact subset and \( \nu_K \) its equilibrium measure. Then \( \nu_{K} \geq \text{cap} K \) on \( \mathbb{C} \) and \( \nu_K = \text{cap} K \) on \( K \setminus F, F \subset \partial K \) a suitable polar \( F \)-set. Moreover, \( \nu_K(z) = \text{cap} K \) for \( z \in \partial K \), whenever \( z \) is regular for the Dirichlet problem for the unbounded component of \( \mathbb{C} \setminus K \).

6. \( \text{cap} \mathbb{B}(z, r) = \text{cap}(\partial \mathbb{B}(z, r)) = r \) and \( \text{cap} K = \text{cap}(\partial K) \leq \text{diam} K \) for any compact set \( K \subset \mathbb{C} \).

For a compact set in the complex plane we introduce its Cauchy transform.

**Definition 3.3.13.** Let \( K \subset \mathbb{C} \) be compact. The function \( f_K : \mathbb{C} \setminus K \to \mathbb{C} \),  
\[
f_K(z) := \begin{cases} 
\int_{K} \frac{d\nu_k(\zeta)}{z-\zeta}, & \text{if } K \text{ is not polar} \\
0, & \text{if } K \text{ is polar}
\end{cases}
\]
is called the Cauchy transform of \( K \). (Recall that \( \nu_K \) is the equilibrium measure of \( K \).)

Obviously, \( f_K \in \mathcal{O}(\mathbb{C} \setminus K) \) and \( f_K|_D \in L^2_k(D) \) for any bounded domain \( D \subset \mathbb{C} \setminus K \). Then:

**Lemma 3.3.14 ([Zwo 2002]).** For a \( \rho \in [0, \frac{1}{2}) \) there exist positive numbers \( C_1, C_2 \) such that for any pair of disjoint compact sets \( K, L \subset \rho E \) and any domain \( D \subset \rho E \setminus (K \cup L) \) the following inequalities hold:
\[
|\langle f_K, f_L \rangle_{L^2_k(D)}| \leq C_2 - C_1 \log \text{dist}(K, L), \quad (3.3.10)
\]
\[
\|f_K\|_{L^2_k(D)}^2 \leq C_2 - C_1 \log(\text{cap} K). \quad (3.3.11)
\]

**Proof.** Obviously, both inequalities are true for any constants \( C_j \) when \( K \) or \( L \) is a polar set. So we may assume that both sets are not polar.

Applying the Fubini theorem, we get the following inequality
\[
|\langle f_K, f_L \rangle_{L^2_k(D)}| = \left| \int_{D} \int_{K} \frac{d\nu_K(\zeta)}{z-\zeta} \int_{L} \frac{d\nu_L(\eta)}{z-\eta} dA_2(z) \right| \leq \int_{K} \int_{L} \int_{\rho E} \frac{1}{|z-\zeta||z-\eta|} dA_2(z) d\nu_L(\eta) d\nu_K(\zeta).
\]

Now we discuss the interior integral.
Take $\zeta, \eta \in \rho E$, $\zeta \neq \eta$. Then
\[
\int_{\rho E} \frac{dA_2(z)}{|z - \zeta||z - \eta|} \leq \int_{E} \frac{dA_2(z)}{|z||z - (\zeta - \eta)|}
\]
\[
= \int_{\frac{1}{2\pi} E} \frac{dA_2(z)}{|z||z - 1|} = \int_{\frac{1}{2\pi} E} \frac{dA_2(z)}{|z||z - 1|} + \int_{\frac{1}{2\pi} E \setminus \frac{1}{2\pi} E} \frac{dA_2(z)}{|z||z - 1|}
\]
Observe that the first term in the last expression is finite and independent of $\eta$ and $\zeta$. For the second summand we proceed as follows:
\[
\int_{\frac{1}{2\pi} E \setminus \frac{1}{2\pi} E} \frac{dA_2(z)}{|z||z - 1|} = \int_{\frac{1}{2\pi} E} \int_{0}^{2\pi} \frac{drd\theta}{|1 - re^{i\theta}|} 
\]
\[
= \int_{\frac{1}{2\pi} E} \int_{0}^{2\pi} \left|1 + \frac{e^{i\theta}}{r} + \frac{e^{2i\theta}}{r^2(1 - e^{i\theta})^2}\right|drd\theta \leq \int_{\frac{1}{2\pi} E} \frac{C_1}{r} \leq -C_1 \log |\zeta - \eta|,
\]
where $C_1$ is independent of the discussed $\zeta$, $\eta$.

Consequently,
\[
\int_{\rho E} \frac{dA_2(z)}{|z - \zeta||z - \eta|} \leq C_2 - C_1 \log |\zeta - \eta|, \quad \zeta, \eta \in \rho E, \quad \zeta \neq \eta,
\]
where $C_1, C_2$ are positive constants.

Coming back to the beginning, we obtain
\[
|\langle fK, fL \rangle_{L^2(D)}| \leq C_2 - C_1 \int_{K} \int_{L} \log |\zeta - \eta|d\nu_K d\nu_L,
\]
which ends the proof. □

The main notion here will be the following potential theoretic function.

**Definition 3.3.15.** Let $D \subset \mathbb{C}$ be a bounded domain. Put $\alpha_D: \overline{D} \rightarrow (-\infty, \infty]$
\[
\alpha_D(z) := \int_{0}^{1/2} \frac{dr}{-r^3 \log(\text{cap}(B(z, r) \setminus D))} = \int_{0}^{1/2} \frac{dr}{-r^3 \log(\text{cap}(\mathbb{B}(z, r) \setminus D))}.
\]

**Remark 3.3.16.** We denote by $A_k(z)$ the annulus with center $z$ and radii $1/2^{k+1}, 1/2^k$, i.e.
\[
A_k(z) := \{w \in \mathbb{C} : 1/2^{k+1} \leq |w - z| \leq 1/2^k\}.
\]

Then for a bounded domain $D \subset \mathbb{C}$ there is an alternative description of $\alpha_D$, namely:
\[
\frac{1}{8} \sum_{k=2}^{\infty} 2^{2k} \leq \alpha_D(z) \leq 8 \sum_{k=1}^{\infty} 2^{2k} \log \text{cap}(A_k(z) \setminus D), \quad z \in \overline{D}.
\]

To get the lower estimate one only has to use the monotonicity of cap, whereas the upper estimate is based on property (4) of cap.

Moreover, $\alpha_D$ is semicontinuous from below on $\overline{D}$ and continuous on $D$; here use properties (4) and (6) of cap and Fatou’s lemma, respectively the Lebesgue theorem.
Remark 3.3.17. For a point \( z_0 = x_0 + iy_0 \in \mathbb{C} \) we define the annuli with respect to the maximum norm, i.e. \( \tilde{A}_k(z_0) := \{ z = x + iy \in \mathbb{C} : 1/2^{k+1} \leq \max\{ |x - x_0|, |y - y_0| \} \leq 1/2^k \} \), where \( k \in \mathbb{N} \). Moreover, let \( \tilde{B}(a, r) := \{ z = x + iy \in \mathbb{C} : \max\{ |x - \text{Re} a|, |y - \text{Im} a| \} < r \} \), where \( a \in \mathbb{C} \) and \( r > 0 \). Then we may define a similar notion to \( \alpha_D \), namely

\[
\tilde{\alpha}_D(z) := \int_0^{1/2} \frac{dr}{-r^3 \log \text{cap}(\tilde{B}(z, r) \setminus D)}, \quad z \in D.
\]

We only note that both functions \( \alpha_D \) and \( \tilde{\alpha}_D \) are comparable and that for the new functions inequalities like the ones in Remark 3.3.16 hold.

It turns out that, in general, the function \( \alpha_D \) is not continuous on \( \overline{D} \) (see the next Example 3.3.18).

Example 3.3.18 ([Zwo 2002]). Now fix \( n \in \mathbb{N} \) and put

\[
M_n := \tilde{A}_n(0) \cap \left\{ \frac{j}{2^n} + i \frac{k}{2^n} : |j|, |k| = 0, \ldots, 2^n - 1 \right\}.
\]

Then \( M_n \) has \( l_n := (2^{1+n} - 1)^2 - (2^n - 1)^2 \) elements. We denote them by \( z_{n,k}, k = 1, \ldots, l_n \).

Then we define the following plane domain

\[
D := \tilde{B}(0, 1/4) \setminus \left( \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{l_n} \tilde{B}(z_{n,k}, r_n) \cup \{0\} \right).
\]

Here the radii \( r_n > 0 \) are chosen such that \( -\log(\text{cap}(\tilde{B}(0, r_n))) = 2n2^{1+n^2} \), \( n \geq 2 \). Observe that the distance between two different \( z_{n,k} \)'s is equal to \( d_n := \frac{1}{2^n} \). Therefore,

\[
d_n - 2r_n \geq d_n - 2 \text{cap}(\tilde{B}(0, r_n)) \geq d_n - 2e^{-n^22^{2n+2n^3}} \\
\geq d_n - 2 \frac{1}{n^22^{2n+2n^3}} = \frac{1}{2^n} \left( 1 - \frac{2}{n^22^{2n+2n^3}} \right) =: b_n \geq \frac{1}{2^n+1} > 0.
\]

Hence two of the different “balls” have a distance which is at least \( b_n \).

In a next step we are going to estimate \( \tilde{\alpha}_D(0) \), namely:

\[
\tilde{\alpha}_D(0) \leq C_1 \sum_{n=1}^{\infty} \frac{2^n}{-\log(\text{cap}(A_n(0) \setminus D))} \leq C_1 \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \frac{2^n}{-\log(\text{cap}(\tilde{B}(z_{n,k}, r_n)))} \\
= C_1 \sum_{n=1}^{\infty} \frac{2^n2^{2n+2n^3}}{n^22^{2n+2n^3}} = 4C_1 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]

Therefore, \( \alpha_D(0) < \infty \).
To see that $\alpha_D$ is not continuous at 0 take an arbitrary point $z \in \tilde{A}_n(0)$. Then
\[
\tilde{\alpha}_D(z) \geq C_2 \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}}{-\log \text{cap}(\tilde{A}_{n+j}(z) \setminus D)}
= C_2 \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}}{\log \frac{1}{2^{1+\alpha^3} \text{cap}(\tilde{A}_{n+j}(z) \setminus D)} + \log 2^{1+n+n^3}}.
\]
To continue with the estimate we note that there exists a $C_3 > 0$ such that
\[
\# \{k = 1, \ldots, l_n : \overline{B}(z_{k,n}, r_n) \subset \tilde{A}_{n+j}(z) \} \geq C_3 2^{2(n^3-j)}, \quad j = 1, \ldots, n^3-1.
\]
Moreover, applying properties (4) and (4') of cap we obtain
\[
\frac{1}{-\log \text{cap}(\tilde{A}_{n+j}(z) \setminus D)} \leq \sum_{k=1}^{l_n} \frac{1}{-\log \text{cap}(\overline{B}(z_{k,n}, r_n))} \leq \frac{2^{2+2n^3}}{n^22^{n+2n^3}} = \frac{4}{n^22n^3}
\]
and
\[
\log^+ \frac{1}{2^{n+1}2^{n^3} \text{cap}(\tilde{A}_{n+j}(z) \setminus D)} \geq \frac{C_3 2^{2(n^3-j)}}{\log^+ \frac{1}{2^{n+1}2^{n^3} \text{cap}(\overline{B}(0,r_n))}}, \quad j = 1, \ldots, n^3-1.
\]
Now, observing that $2^{2(1+n+n^3)} \max \{\text{cap}(\overline{B}(0,r_n)), \text{cap}(\tilde{A}_{n+j}(z) \setminus D)\} < 1$, if $n \geq n_0$ for a suitable $n_0 \in \mathbb{N}$, leads for $n \geq n_0$ to
\[
\tilde{\alpha}_D(z) \geq C_2 \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}}{2\log \frac{1}{2^{1+n+n^3} \text{cap}(\tilde{A}_{n+j}(z) \setminus D)}}
\geq C_2 C_3 \frac{n^3-1}{2} \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}2^{2(n^3-j)}}{n^22^{4n^2+2n^3}} = C_4 \frac{n^3-1}{n^2} \to \infty \quad n \to \infty.
\]
Hence $\lim_{D \uparrow 0} \alpha_D(z) = \lim_{D \uparrow 0} \tilde{\alpha}_D(z) = \infty$.

Finally, we formulate the main result.

**Theorem 3.3.19** ([Zwo 2002]). Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in \partial D$. Then the following properties are equivalent:

(i) $D$ is $\beta$-exhaustive at $z_0$ (i.e. $\lim_{D \uparrow z_0} k_D(z) = \infty$);
(ii) $\lim_{D \uparrow z_0} \alpha_D(z) = \infty$.

**Proof.** For the whole proof we may assume that $D \subset B_E$ and $z_0 = 0 \in \partial D$.

(ii) $\Rightarrow$ (i): Assume that the statement in (i) is not true. Then there exists a sequence $(z_k)_{k \in \mathbb{N}} \subset D \cap B(0,1/8)$ with $\lim_{k \to \infty} z_k = 0$ and $\sup_{k \in \mathbb{N}} k_D(z_k) =: M < \infty$.

Put $K^k_n := A_n(z_k) \setminus D$, $n \geq 2$, $k \in \mathbb{N}$. Since $z_k \in D$ there is an $N_k \in \mathbb{N}$ such that $K^k_n = \emptyset$ for all $k > N_k$. Observe that necessarily $N_k \to \infty$.

By assumption (see Remark 3.3.16) we know that
\[
\frac{1}{8} \alpha_D(z_k) \leq S_k := \sum_{n=2}^{N_k} \frac{2^{2n}}{-\log \text{cap}(K^k_n)}, \quad k \to \infty.
\]
Put

\[ K^k_{n,j} := K^k_n \cap \{ z_k + re^{i\theta} : r > 0, -\pi/3 + (j-1)2\pi/3 \leq \theta \leq \pi/3 + (j-1)2\pi/3 \}, \quad j = 1, 2, 3. \]

In virtue of property (4) of the function \( \text{cap} \), we have

\[
\frac{1}{-\log \text{cap} K^k_n} \leq \sum_{j=1}^{3} \frac{1}{-\log \text{cap} K^k_{n,j}}.
\]

Choose \( j(n, k) \) such that \( \text{cap} K^k_{n,j} \leq \text{cap} K^k_{n,j(n, k)} \) and put \( \tilde{K}^k_n := K^k_{n,j(n, k)} \). Then

\[
\frac{1}{3} \sum_{k=1}^{N_k} \frac{2^{2n_k}}{-\log \text{cap} \tilde{K}^k_n} \rightarrow \infty.
\]

Define

\[
f_{n,k}(z) := \begin{cases} \int_{\tilde{K}^k_n} \frac{d\nu_{n,k}(\zeta)}{z - e^{i\alpha_{n,k} \zeta}} & \text{if } \text{cap} K^k_n \neq 0, \\ 0 & \text{if } \text{cap} K^k_n = 0, \end{cases}, \quad z \in \mathbb{C} \setminus \tilde{K}^k_n,
\]

where \( \nu_{n,k} := \nu_{K^k_n} \) and \( \theta_{n,k} \) such that \( \arg(z_k - e^{i\alpha_{n,k} \zeta}) \in [-\pi/3, \pi/3] \) for all \( \zeta \in \tilde{K}^k_n \). Then

\[
|f_{n,k}(z_k)| \geq \text{Re} \left( \int_{\tilde{K}^k_n} \frac{d\nu_{n,k}(\zeta)}{|z_k - \zeta|} \right) \geq \tilde{C}_3 \int_{\tilde{K}^k_n} \frac{d\nu_{n,k}(\zeta)}{|z_k - \zeta|} \geq C_3 2^{2n},
\]

where \( \tilde{C}_3, C_3 \) are fixed positive constants.

There are two cases to be discussed. The first one: assume that there are a subsequence of \( (z_k) \), denoted again by \( (z_k) \) and a sequence \( (n_k)_k \subset \mathbb{N} \) with \( n_k \leq N_k, \ k \in \mathbb{N} \), such that

\[
\lim_{k \rightarrow \infty} \frac{2^{2n_k}}{-\log \text{cap} \tilde{K}^k_{n_k}} = \infty.
\]

Put \( f_k := f_{n_k,k} \). Then, in virtue of Lemma 3.3.14, we have

\[
\|f_k\|_{L^2(D)}^2 \leq C_2 - C_1 \log \text{cap} \tilde{K}^k_{n_k}.
\]

Therefore, taking (3.3.12) into account, it follows that \( \lim_{k \rightarrow \infty} k_D(z_k) = \infty \); a contradiction.

The second case: we have a certain positive constant \( C_4 \) such that

\[
\frac{2^{2n_k}}{-\log \text{cap} \tilde{K}^k_{n_k}} \leq C_4, \quad k \in \mathbb{N}, \ n = 2, 3, \ldots, N_k.
\]

Put \( c_{k,n} := \text{cap} \tilde{K}^k_n, \ f_{k,n} := f_{\tilde{K}^k_n} \). We are going to choose complex numbers \( a_{k,n} \) with

\[
a_{k,n} f_{k,n}(z_k) \geq 0 \text{ such that if } f_k := \sum_{n=2}^{N_k} a_{k,n} f_{k,n},
\]

then \( \|f_k(z_k)\|_{L^2(D)} \) is unbounded whereas \( \|f_k\|_{L^2(D)} \) remains bounded by a positive constant \( C \). In that situation we have

\[
M \geq k_D(z_k) \geq \frac{|f_k(z_k)|^2}{\|f_k\|_{L^2(D)}^2} \geq \frac{1}{C} |f_k(z_k)|^2;
\]
a contradiction.

In a first step observe the following inequalities (see Lemma 3.3.14):

\[ |2 \Re(f_{k,m}, f_{k,n})_L^2(D)| \leq \|f_{k,m}\|_L^2(D) + \|f_{k,n}\|_L^2(D) \leq 2C_2 - C_1 \log(c_{k,m}c_{k,n}), \]

when \(|n - m| \leq 1\), and

\[ |2 \Re(f_{k,m}, f_{k,n})_L^2(D)| \leq 2C_2 + 2C_1 \max |\log\left(\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right)|, \quad |\log\left(\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}\right)| \leq 2C_2 + C_3 \nu, \]

when \(|n - m| \geq 2\).

Put \(a_{k,n} := 0\) if \(\text{cap} \, K_{k,n} = 0\). Then

\[ \|f_k\|_L^2(D) \leq C_7 \left( \sum_{n=2}^{N_k} |a_{k,n}|^2 (-\log \, c_{k,n}) + C_6 \sum_{n,m=2, |n-m| \geq 2} |a_{k,n}| |a_{k,m}| n m \right) \]

\[ \leq C_7 \left( \sum_{n=2}^{N_k} |a_{k,n}|^2 (-\log \, c_{k,n}) + \left( \sum_{n=2}^{N_k} n |a_{k,n}| \right)^2 \right). \]

Let \(|a_{k,n}| := -\frac{2^n}{\log c_{k,n}} b_{k,n}\), where the numbers \(b_{k,n} \geq 0\) will be fixed later. Then

\[ |f_k(z_k)| = \sum_{n=2}^{N_k} a_{k,n} f_{k,n}(z_k) \geq C_3 \sum_{n=2}^{N_k} \frac{2^n}{\log c_{k,n}} b_{k,n} 2^n. \]

So we have to look for numbers \(b_{k,n}\) such that \(|f_k(z_k)| \underset{k \to \infty}{\longrightarrow} \infty\), but

\[ \|f_k\|_L^2(D) \leq C_7 \left( \sum_{n=2}^{N_k} \frac{2^n}{\log c_{k,n}} (b_{k,n})^2 + \left( \sum_{n=2}^{N_k} \frac{n}{2^n} \log c_{k,n} b_{k,n} \right)^2 \right) \quad (3.3.13) \]

remains bounded.

Put \(\nu_{k,n} := -\frac{2^n}{\log c_{k,n}}, \, k \in \mathbb{N}\). Recall that \(S_k = \sum_{n=2}^{N_k} \nu_{k,n} \underset{k \to \infty}{\longrightarrow} \infty, \, C_4, \, k \in \mathbb{N}\), and \(N_k \underset{k \to \infty}{\longrightarrow} \infty\). So we may find sequences \((q_{k,j})_{j=0}^{q_k}\), where \(q_{k,0} = 1, \, q_{k,q_k} = N_k\), and \(q_k \underset{k \to \infty}{\longrightarrow} \infty\) such that

\[ \nu_{k,n_{k,j}+1} + \cdots + \nu_{k,n_{k,j+1}} > 1, \quad \frac{1}{2^j} < \frac{1}{j+1}, \quad j = 0, \ldots, q_k - 1, \quad l > n_{k,j+1}. \]

Now we take

\[ b_{k,n_{k,j}+1} = \cdots = b_{k,n_{k,j+1}} := \frac{1}{(j+1)( \nu_{k,n_{k,j}+1} + \cdots + \nu_{k,n_{k,j+1}} )}, \quad j = 0, \ldots, q_k - 1. \]

With this setting we finally obtain that \(|f_k(z_k)| \underset{k \to \infty}{\longrightarrow} \infty\) and that \(|\|f_k\|_L^2(D)\|_k\) remains bounded (compare (3.3.13)). So this part of the proof is complete.
3.3. Bergman exhaustiveness

(i) \implies (ii): Suppose that there is a sequence \((z_k)_k \subset D, z_k \to 0\), such that, for a suitable positive number \(M, \alpha_D(z_k) \leq M\) for all \(k\). Then

\[
\sum_{n=2}^{\infty} \frac{2^{2n}}{\log \text{cap}(A_n(z_k) \setminus D)} \leq 8M.
\]

In particular, if \(c_{k,n} := \text{cap}(A_n(z_k) \setminus D)\) then \(\log c_{k,n} \leq - \frac{2^{2n}}{8M}\), \(k, n \in \mathbb{N}, n \geq 2\), and therefore we may find an \(n_0 \in \mathbb{N}\) such that \(\log c_{k,n} + 1 < -(n + 1) \log 2 - 1, n > n_0, k \in \mathbb{N}\).

Let \(z \in A_n(z_k), 1 \leq n < n_0\), then

\[
\frac{1}{2} + \frac{1}{2^n + 1} \geq |z - z_k| + |z_k| \geq |z| \geq |z - z_k| - |z_k| \geq \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}},
\]

when \(|z_k| < \frac{1}{2^n + 1}\), i.e. for any \(k \geq k_0, k_0\) suitably chosen. Choose a domain \(D' \supset D\) such that \(D' \cap \mathbb{B}(0, \frac{1}{2^n + 1}) = D' \cap \mathbb{B}(0, \frac{1}{2^n + 1})\), such that \(A_n(z_k) \setminus D' = \varnothing, 1 \leq n < n_0, k \geq k_0\).

Applying the localization result (cf. Theorem 3.1.5) for the Bergman kernel, we still know that \(\lim_{k \to \infty} k_N(z_k) = \infty\).

Now, fix a \(k \geq k_0\). Recall that there is an \(n_1 > 2n_0\) such that \(\mathbb{B}(z_k, \frac{1}{2^n + 1}) \subset D'\). We exhaust \(D'\) by a sequence of domains \(D_j' \subset D'\) with real analytic boundaries such that \(\sum_{n=2}^{\infty} \frac{2^{2n}}{\log \text{cap}(A_n(z_k) \setminus D_j')} < 8M, \partial(A_2(z_k) \setminus D_j') = \partial \mathbb{B}(z_k, 1/4), \tilde{K}_n := A_N(z_k) \setminus D_j'\) is either empty or non polar, and any boundary point of \(\tilde{K}_n\), if \(\tilde{K}_n \neq \varnothing\), is a regular point with respect to the unbounded component of its complement. So Frostman’s theorem (see Theorem 3.3.4 in [Ran 1995]) together with the continuity principal for logarithmic potentials (see Theorem 3.1.3 in [Ran 1995]) implies the logarithmic potential \(p_n := p_{\tilde{K}_n}\), if \(n \geq 3\) and \(\tilde{K}_n \neq \varnothing\), is continuous on \(C\). For an \(n \geq 3\) such that \(\tilde{K}_n = \varnothing\) put \(p_n := -\infty\).

For \(n \geq 3\) choose \(\chi_n \in C^\infty(\mathbb{R}, [0, 1])\) such that \(\chi_n = 0\), if \(\tilde{K}_n = \varnothing\), or

\[
\chi_n(t) := \begin{cases} 1, & \text{if } t \leq \log \text{cap} \tilde{K}_n + 1/2 \\ 0, & \text{if } t \geq -(n + 1) \log 2 - 1/2, \end{cases}
\]

and \(|\chi_n(t)| \leq \frac{2}{M_1 \log \text{cap} \tilde{K}_n}\) where \(M_1\) is a suitable positive number.

For \(n \geq 3\) define \(f_n := f_{\tilde{K}_n}\) and \(\varphi_n := \chi_n \circ p_n\). Note that if \(\tilde{K}_n \neq \varnothing\) then \(p_n(z) \geq -(n + 1) \log 2, z \notin A_{n-1}(z_k) \cup A_n(z_k) \cup A_{n+1}(z_k)\), and that \(p_n \in C^\infty(\mathbb{C} \setminus \tilde{K}_n)\). So \(\varphi_n\) is a smooth function with support in \(A_{n-1}(z_k) \cup A_n(z_k) \cup A_{n+1}(z_k)\) such that \(\varphi_n|_{\tilde{K}_n} = 1\) and \(\frac{\partial \varphi_n}{\partial z}(z), z \notin \tilde{K}_n\).

For \(n = 2\) we put \(p_2(z) := \log |z|\) and take a \(\chi_2 \in C^\infty(\mathbb{R}, [0, 1])\) such that

\[
\chi_2(t) := \begin{cases} 0, & \text{if } t \leq -\log 8 \text{ or } t \geq -\log 2 \\ 1, & \text{if } t \text{ is near } -\log 4, \end{cases}
\]

and \(|\chi_2(t)| \leq \frac{2}{\log 4}\). Again, let \(\varphi_2 := \chi_2 \circ p_2\) and put \(f_2 := 1\).

Then

\[
\left| \frac{\partial \varphi_n}{\partial z}(z) \right| \leq \frac{|f_n(z)|}{-M_2 \log \text{cap} \tilde{K}_n}, z \in \frac{1}{2} E \setminus \tilde{K}_n, n \geq 2.
\]
Finally, we define
\[ \varphi := \sup \{ \varphi_n : n \geq 2 \}. \]
Note that the supremum is taken over at most three functions. \( \varphi \) is a Lipschitz function satisfying \( \varphi|_{\partial D} = 1 \) and \( \varphi = 0 \) in a neighborhood of \( z_k \).

Now let \( f \in L^2_b(D') \). Then the Cauchy formula and the Green formula lead to the following equations:
\[
|f(z_k)| = \frac{1}{2\pi} \left| \int_{\partial D_j} \frac{f(\lambda)}{\lambda - z_k} d\lambda \right| = \frac{1}{2\pi} \left| \int_{\partial D_j} \frac{f(\varphi)(\lambda)d\lambda}{\lambda - z_k} \right| = \frac{1}{\pi} \int_{D_j} \frac{f(\lambda) \partial \varphi}{\lambda} dA_2(\lambda).
\]
Applying various versions of the Schwarz inequalities and Lemma 3.3.14 finally gives the following inequalities
\[
|f(z_k)| \leq M_4 \sum_{n=2}^{\infty} \left( \int_{A_n(z_k) \setminus K_n} |f(\lambda)| \left( \frac{|f_{n-1}(\lambda)|}{-\log \text{cap } K_{n-1}} + \frac{|f(\lambda)|}{-\log \text{cap } K_n} + \frac{|f_{n+1}(\lambda)|}{-\log \text{cap } K_{n+1}} \right) dA_2(\lambda) \right)^{1/2}
\]
\[
\leq M_5 \left( \sum_{n=2}^{\infty} \frac{1}{-\log \text{cap } K_n} \right)^{1/2} \left( \sum_{n=2}^{\infty} \frac{2^n}{-\log \text{cap } K_n} \right)^{1/2} \leq \sqrt{M} \|f\|_{D'}. \]
where the constant on the right side is independent of \( k \). This estimate is true for all sufficiently large \( k \). Therefore, \((k_D(z_k))_k\) is bounded; a contradiction. \( \square \)

Remark 3.3.20. There are similar considerations as in Theorem 3.3.19 for the so called point evaluation. To be more precise, let \( z_0 \in \partial D \), where \( D \subset \mathbb{C} \) is a bounded domain. Recall that \( V := \{ f \in L^2_b(D) : f \text{ is holomorphic in } D \cup \{ z_0 \} \} \) is dense in \( L^2_b(D) \) (cf. Theorem 3.1.2). Therefore, we may define the evaluation functional on \( V \), i.e. \( \Phi_{z_0} : V \to \mathbb{C}, \Phi_{z_0}(f) := f(z_0) \). The point \( z_0 \) is called to be a bounded evaluation point for \( L^2_b(D) \) if \( \Phi \) extends to a continuous functional on \( L^2_b(D) \). There is the following description of such points [Hed 1972].

Theorem. Let \( D \) and \( z_0 \) be as above. Then \( \alpha_D(z_0) = \infty \) iff \( z_0 \) is not a bounded evaluation point for \( L^2_b(D) \).

Observe, if \( z_0 \) is not a bounded evaluation point then \( D \) is \( b \)-exhaustive at \( z_0 \). Nevertheless, the converse statement is false (see Example 3.3.18).

Remark 3.3.21. For a bounded domain \( D \subset \mathbb{C} \) there are analogous notions like the Bergman kernel taking derivatives into account, namely the \( n \)-th Bergman kernel
\[
k^{(n)}_D(z) := \sup \{ |f^{(n)}(z)|^2 : f \in L^2_b(D) \setminus \{ 0 \}, \|f\|_{L^2_b(D)} = 1 \}, \quad n \in \mathbb{N}_0, \quad z \in D.
\]
Observe that \( k_D = k^{(0)}_D \). Moreover, one has the following potential theoretic function
\[
\alpha^{(n)}_D(z) := \int_0^{1/2} \frac{dr}{r^{2n+3}(-\log \text{cap } (B(z, r) \setminus D))}, \quad z \in \overline{D}, \quad n \in \mathbb{N}_0.
\]
3.3. Bergman exhaustiveness

Observe that $\alpha_D = \alpha_D^{(0)}$. There is the following relation between these notions (see [Pfl-Zwo 2003a])

**Theorem.** Let $n \in \mathbb{N}_0$ and $d > 1$. Then there is a $C > 0$ such that

- for any domain $D \subset \mathbb{C}$ with $\text{diam } D < d$
  
  $$C\alpha_D^{(n)}(z) \leq k_D^{(n)}(z), \quad z \in D;$$

- for any domain $D \subset \mathbb{C}$ with $\frac{1}{d} < \text{diam } D < d$
  
  $$k_D^{(n)}(z) \leq C \max\{1, \alpha_D^{(n)}(z)(\log \alpha_D^{(n)}(z))^2\}, \quad z \in D.$$

\[?\] Let $D \subset \mathbb{C}$ be a domain and $z_0 \in \partial D$. Is it true that $\lim_{D \ni z_0} k_D^{(n)}(z) = \infty$ implies that $\lim_{D \ni z_0} \alpha_D^{(n)}(z) = \infty$?

With the help of the above theorem there is a complete description of those Zalcman domains which are $b$–exhaustive at all of its boundary points.

**Corollary 3.3.22 ([Juc 2003]).** Let

$$D := E \setminus \left( \bigcup_{k=1}^{\infty} \mathbb{B}(x_k, r_k) \cup \{0\} \right)$$

be a Zalcman domain \(^7\), where $x_k > x_{k+1} > 0$, $\lim_{k \to \infty} x_k = 0$, $r_k > 0$ with $\mathbb{B}(x_k, r_k) \subset E$, $\mathbb{B}(x_k, r_k) \cap \mathbb{B}(x_j, r_j) = \emptyset$, $k, j \geq 1$, $k \neq j$. Assume that

$$\exists \theta_1 \in (0, 1) \exists \theta_2 \in (\Theta_1, 1) : \quad \Theta_1 \leq \frac{x_{k+1}}{x_k} \leq \Theta_2, \quad k \in \mathbb{N}.$$

Then $D$ is $b$–exhaustive iff $D$ is $b$–exhaustive at 0 iff $\sum_{k=1}^{\infty} \frac{1}{x_k \log x_k} = \infty$ iff $\alpha_D(0) = \infty$.

Observe that special cases were treated also in [Ohs 1993] and [Che 1999]. Moreover, we mention that the domains $D$ in Corollary 3.3.22 are fat domains, but not all of them are $b$–exhaustive (for another example see [Jar-Pfl-Zwo 2000]).

**Proof.** First, observe that for every boundary point $z_0$ except the origin we have

$$\lim_{D \ni z_0} k_D(z) = \infty$$

(see Theorem 6.1.17 in [J-P 1993]).

Obviously, $\mathbb{B}(x_{k+1} - r_{k+1}/2, r_{k+1}) \subset \mathbb{B}(0, \delta) \setminus D$, $\delta \in (x_{k+1}, x_k)$. Then

$$\alpha_D(0) \geq \sum_{k=1}^{\infty} \int_{x_{k+1}}^{x_k} \frac{dr}{r \log \text{cap}(\mathbb{B}(0, r) \setminus D)}$$

$$\geq \sum_{k=k_0}^{\infty} \int_{x_{k+1}}^{x_k} \frac{dr}{r \log \frac{r_{k+1}}{r_k}} \geq \sum_{k=k_0}^{\infty} \frac{1}{x_k \log \frac{r_{k+1}}{r_k}} \geq C \sum_{k=k_0}^{\infty} \frac{1}{x_k^2 \log r_{k+1}},$$

where $C$ is a constant. Observe that for the last inequality the assumption on the centers $x_k$ was used.

\(^7\) Observe that we use here a slightly more general notion than the one of a Zalcman type domain in Section 2.7.
Now, the divergence of the series in the corollary implies that \( \alpha_D(0) = \infty \). In virtue of the lower semicontinuity of the function \( \alpha_D \) it follows that \( \lim_{D \ni z \to 0} \alpha_D(z) = \infty \).

On the other hand we have

\[
\alpha_D(0) = \left( \int_{x_1}^{1/2} + \sum_{k=1}^{\infty} \int_{x_k}^{x_{k+1}} \right) -r^3 \log \text{cap}(B((0,r) \setminus D)) dr
\leq C_1 + \sum_{k=1}^{\infty} \frac{x_k - x_{k+1}}{x_k^{3+1}} \sum_{j=k}^{\infty} \frac{-1}{\log r_j} \leq C_1 + C_2 \sum_{j=1}^{\infty} \frac{-1}{\log r_j} \sum_{k=1}^{j} \frac{1}{x_k}
\leq C_1 + C_2 \sum_{j=1}^{\infty} \frac{-1}{\log r_j} \sum_{k=1}^{j} \frac{\Theta_2^{2(j-k)}}{x_j^2} \leq C_1 + C_3 \sum_{j=1}^{\infty} \frac{-1}{x_j^2 \log r_j},
\]

where \( C_1 \geq 0 \) and \( C_2, C_3 > 0 \) are suitable numbers. Observe that the last three inequalities follow from the assumptions on the centers \( x_k \).

If the series in the corollary does converge, then \( \alpha_D(0) < \infty \). Moreover, directly from the definition we see that \( \alpha_D \) restricted to the interval \((-1/4,0)\) is monotonically increasing. Hence \( \limsup_{D \ni z \to 0} \alpha_D(x) \leq \alpha_D(0) < \infty \). So, the corollary is proved. \( \square \)

**Example 3.3.23.** We discuss the particular case of a Zalcman domain, namely \( x_k := (1/2)^k \) and \( r_k := (1/2)^{kN(k)} \), where \( N_k \in \mathbb{N}, k \geq 2 \). Then we have

\[
D \text{ is } b\text{-exhaustive} \iff \sum_{k=2}^{\infty} \frac{2^k}{kN(k) \log 2} = \infty.
\]

On the other hand, following Ohsawa [Ohs 1993] we have

\[
D \text{ is hyperconvex} \iff \sum_{k=2}^{\infty} \frac{1}{N(k)} = \infty.
\]

So we see that there are plenty of Zalcman domains which are not hyperconvex but, nevertheless, they are \( b\)-exhaustive.

### 3.4. \( L^2_b \)-domains of holomorphy

The boundary behavior of the Bergman kernel may be used to give a complete description of \( L^2_b \)-domains of holomorphy \(^8\). The precise result is the following one.

**Theorem 3.4.1 ([Pfl-Zwo 2002]).** For a bounded domain \( D \subset \mathbb{C}^n \) the following conditions are equivalent:

(i) \( D \) is an \( L^2_b \)-domain of holomorphy;

(ii) \( \limsup_{D_{D \ni z \to z_0}} k_D(z) = \infty \) for every boundary point \( z_0 \in \partial D \).

\(^8\) Recall that a domain \( D \subset \mathbb{C}^n \) is an \( L^2_b \)-domain of holomorphy if for any pair of open sets \( U_1, U_2 \subset \mathbb{C}^n \) with \( \emptyset \neq U_1 \subset D \cap U_2 \neq U_2, U_2 \) connected, there is an \( f \in L^2_b(G) \) such that for any \( F \in \mathcal{O}(U_2) \): \( f|_{U_1} \neq F|_{U_1} \).
3.4. $L^2_n$–domains of holomorphy

Remark 3.4.2. There is also the following more geometric condition which is equivalent to (i) of Theorem 3.4.1:
(iii) for any boundary point $z_0 \in \partial D$ and for any open neighborhood $U = U(z_0)$ the set $U \setminus D$ is not pluripolar \footnote{Recall that a set $P \subset \mathbb{C}^n$ is called to be pluripolar if there is a $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $P \subset u^{-1}(\infty)$.}.

Proof of Theorem 3.4.1. The case $n = 1$ may be found in [Con 1995]. So we will always assume that $n \geq 2$.

(ii) $\implies$ (i): Suppose that $G$ is not an $L^2_n$–domain of holomorphy. Then there are concentric polydiscs $P \subset \hat{P}$ satisfying $P \subset D$, $\partial P \cap \partial D \neq \varnothing$, and $\hat{P} \not\subset D$ such that for any function $g \in L^2_n(D)$ there exists an $\hat{g} \in \mathcal{H}^\infty(\hat{P})$ with $\hat{g}|_P = g|_P$.

Let $a$ be the center of $P$ and let $L$ be an arbitrary complex line through $a$. Then $L \cap \hat{P} \setminus D =: K$ is a polar set (in $L$).

Indeed, suppose that $K$ is not polar. Fix a compact non–polar subset $K' \subset K$. Then, according to Theorem 9.5 in [Con 1995], there is a non–trivial function $f \in L^2_n(L \setminus K')$ which has no holomorphic extension to $L$. Since $K' \cap D = \varnothing$, Theorem 3.1.1 guarantees the existence of a function $F \in L^2_n(D)$ with $F|_{L \cap D} = f|_{L \cap D}$. Hence, we find $\hat{F} \in \mathcal{O}(\hat{P})$ such that $\hat{F}|_P = F|_P$. In particular, $\hat{F}|_{L \cap \hat{P}}$ extends $f$ to the whole of $L$; a contradiction.

So $L \cap \hat{P} \setminus D$ is connected \footnote{Recall that for a plane domain $G$ and a relatively closed polar subset $M \subset G$ the open set $G \setminus M$ is connected.}. Since $L$ is arbitrary, $D \cap \hat{P}$ is connected. Therefore, for any function $g \in L^2_n(D)$ there exists a unique holomorphic extension $\hat{g} \in \mathcal{H}^\infty(\hat{P})$ with $\hat{g}|_{D \cap \hat{P}} = g|_{D \cap \hat{P}}$.

Consider the linear space

$$A := \{(g, \hat{g}) : g \in L^2_n(D)\} \subset L^2_n(D) \times \mathcal{H}^\infty(\hat{P})$$

equipped with the norm $\|(g, \hat{g})\| := \|g\|_{L^2_n(D)} + \|\hat{g}\|_{\mathcal{H}^\infty(\hat{P})}$. Then $A$ is a Banach space.

Observe that the mapping $A \ni (g, \hat{g}) \mapsto g \in L^2_n(D)$ is a one-to-one, surjective, continuous, linear mapping. Hence, in view of the Banach open mapping theorem, its inverse map is also continuous, i.e. there is a $C > 0$ such that

$$\|(g, \hat{g})\| \leq C\|g\|_{L^2_n(D)},\quad g \in L^2_n(D).$$

In particular, $\|\hat{g}\|_{\mathcal{H}^\infty(\hat{P})} \leq C\|g\|_{L^2_n(D)}$. So we are led to the following estimate

$$\sup\{k_D(z) : z \in D \cap \hat{P}\} = \sup\left\{\frac{|g(z)|^2}{\|g\|^2_{L^2_n(D)}} : z \in D \cap \hat{P},\ 0 \neq g \in L^2_n(D)\right\} \leq C^2.$$

In particular, $\limsup_{z \to w} k_D(z) \leq C^2$ for a point $w \in \partial P \cap \partial D \neq \varnothing$ (recall that such a point exists); a contradiction.

Before we are able to start the proof of (i) $\implies$ (ii) we need some auxiliary results.

Lemma 3.4.3. Let $G \subset \mathbb{C}$ be a bounded domain and let $a \in \partial G$. Assume that $\limsup_{z \to w} k_G(z) < \infty$. Then there is a neighborhood $U = U(a)$ such that $U \setminus G$ is polar.
Proof. Suppose that Lemma 3.4.3 is not true.

First we claim that for any \( r > 0 \) the intersection \( \mathbb{B}_1(a, r) \cap \partial G \) is not polar. Otherwise, there is an \( r_0 > 0 \) such that \( \mathbb{B}_1(a, r_0/4) \cap \partial G \) is polar. Observe that \( \mathbb{B}_1(a, r_0/4) \setminus G \) is not polar. Therefore, there exists a \( b_0 \in \mathbb{B}_1(a, r_0/4) \setminus \overline{G} \). Choose a point \( b \in \mathbb{B}_1(a, r_0/4) \cap G \). Since \( \mathbb{B}_1(a, r_0) \cap \partial G \) is polar, there exists an \( s \in (0, r_0/2) \) such that \( \partial \mathbb{B}_1(b_0, s) \cap \partial G = \emptyset \), \( \partial \mathbb{B}_1(b_0, s) \cap G \neq \emptyset \), and \( \partial \mathbb{B}_1(b_0, s) \subset \mathbb{B}_1(a, r_0) \). Hence, \( \partial \mathbb{B}_1(b_0, s) \subset G \). Therefore, for any \( z \in \partial \mathbb{B}_1(b_0, s) \), one has \( [b_0, z] \cap \partial G \neq \emptyset \). Then, in virtue of Theorem 5.1.7 in [Arm-Gar 2001], it follows that \( \partial \mathbb{B}_1(b_0, s) \) is a polar set; a contradiction.

Hence, there is a sequence \( (a_j)_j \subset \partial G \), \( a_j \rightarrow a \), where all the points \( a_j \) are regular boundary points of \( G \). Suppose for a moment that we knew that \( \lim_{G \ni z \rightarrow b} k_G(z) = \infty \) for a regular boundary point \( b \) of \( G \). Then \( \lim \sup_{G \ni z \rightarrow b} k_G(z) = \infty \) which obviously contradicts the assumption of Lemma 3.4.3.

It remains to prove the following claim: Let \( b \in \partial G \) be a regular point. Then \( k_G(z) \rightarrow \infty \) when \( G \ni z \rightarrow b \).

Here we will use the following relation of the Bergman kernel and the Azukawa metric given in [Ohs 1995], namely, there is a \( C > 0 \) such that

\[
\sqrt{k_G(z)} \geq CA_G(z; 1), \quad z \in G.
\]

Define \( G_p := \{ z \in G : \log g_G(p, z) < -1 \} \), \( p \in G \), and \( r(p) := \text{diam} G_p \). Then, using [Zwo 2000c], we get

\[
A_G(p, 1) = eA_{G_p}(p, 1) \geq eA_{\mathbb{B}_1(p, r(p))}(p, 1) = \frac{e}{r(p)} \rightarrow \infty \text{ as } p \rightarrow b.
\]

So it remains to show that \( r(p) \rightarrow 0 \) when \( p \rightarrow b \). Suppose this is not true. Then we find an \( \varepsilon > 0 \), sequences \( G \ni p_j \rightarrow b \) and \( G \ni z_j \rightarrow z^* \in \overline{G} \) such that \( |p_j - z_j| \geq \varepsilon \) and \( \log g_G(p_j, z_j) < -1 \), \( j \in \mathbb{N} \). Choose a small disc \( V \) around \( z^* \) with \( b \notin V \). Then we have \( g_G(p_j, z_j) \geq g_{\overline{G}}(p_j, z_j) \), where \( G := G \cup V \). Now, observe that \( g_{\overline{G}}(p_j, \cdot) = g_{\overline{G}}(\cdot, p_j) \) and that \( \log g_{\overline{G}}(p_j, \cdot) \rightarrow 0 \) pointwise. Applying that \( \log g_{\overline{G}}(p_j, \cdot) \) are harmonic functions, the Vitali theorem implies that \( \log g_{\overline{G}}(p_j, \cdot) \) tends uniformly to 0 on some small neighborhood of \( z^* \) contradicting that \( \log g_G(p_j, z_j) < -1 \) for all \( j \).

\[ \square \]

Lemma 3.4.4. Let \( D \subset \mathbb{C}^n, n \geq 2, \) be a domain and let \( 0 < r < t \). For any \( z' \in \mathbb{C}^{n-1} \) define

\[ D_{z'} := \{ z_n \in tE : (z', z_n) \in D \} =: tE \setminus K(z'). \]

Assume that \( K(0') \) is polar and that there is a neighborhood \( V \) of \( 0' \) such that for almost all \( z' \in V \) the set \( K(z') \) is also polar.

Then there is a neighborhood \( V' \subset V \) of \( 0' \) such that for any \( f \in L_h^2(D) \) there exists an \( \tilde{f} \in \mathcal{O}(V' \times rE) \) with \( f = \tilde{f} \) on \( D \cap (V' \times rE) \).

Proof. Since \( K(0') \) is a polar set, there is an \( s \) with \( r < s < t \) such that \( K(0') \cap \partial B_1(0, s) = \emptyset \). Therefore, we find a neighborhood \( V' = V'(0') \subset V \) such that \( K(z') \cap \partial B_1(0, s) = \emptyset \), \( z' \in V' \). Then we may define

\[
\tilde{f}(z', z_n) := \frac{1}{2\pi} \int_{\partial B_1(0, s)} \frac{f(z', \lambda)}{\lambda - z_n} d\lambda, \quad (z', z_n) \in V' \times B_1(0, s).
\]

Obviously, \( \tilde{f} \in \mathcal{O}(V' \times B_1(0, s)) \).
On the other hand, using that \( f \in L^2_h(D) \), the Fubini theorem and the assumptions made in Lemma 3.4.4 give that for almost all \( z' \in V' \) the function \( f(z',\cdot) \in L^2_h(\mathbb{B}_1(0,t) \setminus K(z')) \) and \( K(z') \) is polar. Hence, \( f(z',\cdot) \) extends to a holomorphic function on \( \mathbb{B}_1(0,t) \) for almost all \( z' \in V' \). Applying the Cauchy integral formula, we obtain \( f(z',z_n) = \hat{f}(z',z_n), (z',z_n) \in V' \times \mathbb{B}_1(0,s) \), for almost all \( z' \in V' \). Since this set is dense in \((V' \times \mathbb{B}_1(0,s)) \cap D \), we have reached the claim in Lemma 3.4.4.

Now we are able to complete the proof of Theorem 3.4.1.

(i) \( \implies \) (ii): Fix a boundary point \( w \in \partial D \). First we discuss the case when \( w \notin \text{int}(\mathcal{D}) \). Then there is a sequence \((z_j)_j \subset \mathbb{C}^n \) such that \( z_j \longrightarrow w \) and \( z_j \notin \mathcal{D}, j \in \mathbb{N} \). By \( r_j \) we denote the largest radius such that \( B_j := \mathbb{B}_n(z_j,r_j) \) does not intersect \( \mathcal{D} \). Select \( w_j \in \partial B_j \cap \partial D \). Then, \( w_j \longrightarrow w \). Observe that the domain \( D \) satisfies the general outer cone condition at \( w \) (see Theorem 6.1.17 in [J-P 1993]). Therefore, \( \lim_{D \ni z \to w} k_D(z) = \infty \). Hence, (ii) follows.

From now on we assume that \( w \in \text{int}(\mathcal{D}) \). Suppose that (ii) is not true for \( w \). Then there are a polydisc \( P \subset \mathcal{D} \) with the center at \( w \) and a constant \( C > 0 \) such that

\[
k_D(z) \leq C, \quad z \in D \cap P.
\]

Now, let \( L \) be a complex line through \( P \). Then \( (L \cap P) \setminus D \) is a polar set (in \( L \)) or it is empty. Indeed, otherwise we apply Lemma 3.4.3. Therefore,

\[
\sup\{k_D(z) : z \in L \cap P \cap D\} = \infty.
\]

Then, in virtue of Theorem 3.1.1, it follows that \( \sup\{k_D(z) : z \in L \cap D \cap P\} = \infty \); a contradiction.

Observe that there is a complex line \( L^* \) passing through \( w \) and \( P \cap D \). We may assume that \( w = 0 \) and, after a linear change of coordinates, that \( P = E^n \) and \( L^* = \{(0,\ldots,0)\} \times \mathbb{C} \). So the assumptions of Lemma 3.4.4 are fulfilled with respect to some neighborhood \( V \subset E^{n-1} \) of \( 0' \in \mathbb{C}^{n-1} \). Therefore, there is a neighborhood \( V' = V'(0') \subset V \) such that for any \( f \in L^2_h(D) \) there is an \( f = f \in \mathcal{O}(V' \times \mathbb{B}_1(0,1/2)) \) with \( f = f \) on \( D \cap (V' \times \mathbb{B}_1(0,1/2)) \) contradicting the assumption in (i).

\[\square\]

**Remark 3.4.5.** In [Irg 2003], the following generalization of Theorem 3.4.1 may be found.

**Theorem.** Let \((X,\pi)\) be a Riemann domain over \( \mathbb{C}^n \) such that \( \pi(X) \) is bounded. Let \((\hat{X},\hat{\pi})\) be the envelope of holomorphy and \((\hat{X},\hat{\pi})\) the \( L^2_h(X) \)-envelope of holomorphy of \((X,\pi)\). Then \((\hat{X},\hat{\pi})\) embeds into \((\hat{X},\hat{\pi})\) and the difference of these two sets is a pluripolar subset of \( \hat{X} \).

Applying Theorem 3.3.9, Theorem 3.4.1 may be used to get the following result.

**Corollary 3.4.6** ([Jar-Pfl 1996]). *Any bounded balanced domain of holomorphy is an \( L^2_h \)-domain of holomorphy.*

\[\text{\footnotesize (11)}\] Recall that a relatively closed polar subset of a plane domain is a removable set of singularities for square–integrable holomorphic functions.
It is an open problem to characterize those unbounded domains of holomorphy that are $L^2_h$-domains of holomorphy. Even more, so far there is no description of such unbounded domains that carry a non-trivial $L^2_h$-function.

3.5. Bergman completeness

Let $G \subset \mathbb{C}^n$ be a domain such that $k_G(z) > 0$, $z \in G$. Then the Bergman kernel $k_G$ is a logarithmically psh function on $G$. So

$$\beta_G(z;X) := \left( \sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log k_G(z) X_j X_k \right)^{1/2}, \quad z \in G, \ X \in \mathbb{C}^n,$$

gives a hermitian pseudometric on $G$. It is the Bergman pseudometric. Recall that there is another description of the Bergman pseudometric. Let $G \subset \mathbb{C}^n$ be as above. We define

$$M_G(z;X) = \sup \left\{ |f'(z)X| : f \in L^2_h(G), \|f\|_{L^2_h(G)} = 1, \ f(z) = 0 \right\}, \ z \in G, \ X \in \mathbb{C}^n.$$

Then,

$$\beta_G(z;X) = \frac{M_G(z;X)}{\sqrt{k_G(z)}}. \quad (3.5.14)$$

Observe that $\beta_G$ is a metric if

$$\forall z \in G \quad \text{and} \quad \forall X \in \mathbb{C}^n, \ X \neq 0 \ \exists g, f \in L^2_h(G) : g(z) \neq 0 \ \text{and} \ f(z) = 0, \ f'(z)X \neq 0. \quad (3.5.15)$$

The Bergman pseudodistance on $G$ is given by

$$b_G(z,w) := \inf \left\{ \int_0^1 \beta_G(\gamma(t);\gamma'(t))dt : \gamma \in C^1([0, 1], G) : \gamma(0) = z, \ \gamma(1) = w \right\}, \ z, w \in G.$$

Under the condition (3.5.15), the function $b_G$ is in fact a distance.

One of the main questions here is to decide which domain in $\mathbb{C}^n$ is $b_G$-complete.

**Definition 3.5.1.** A domain $G \subset \mathbb{C}^n$ satisfying $k_G(z) > 0$ for all $z \in G$ is called Bergman-complete (for short $b$-complete or $b_G$-complete) if $b_G$ is a distance and if for any $b_G$-Cauchy sequence $(z_j)_j \subset G$ there is a point $a \in G$ such that $\lim_{j \to \infty} z_j = a$.

Obviously, for any bounded domain $D \subset \mathbb{C}^n$, $\beta_D$ is a metric and $b_D$ is a distance on $D$. For a not necessarily bounded domain $D$, we have the following sufficient criterion (see [Che-Zha 2002]).

**Theorem 3.5.2.** Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain (not necessarily bounded). Assume for any point $w \in D$ there is an $r > 0$ such that

$$A_w(D; r) := \{ z \in D : \log g_D(w, z) < -r \} \subset D. \quad (13)$$

Then $\beta_D$ is a metric on $D$ and $b_D$ is a distance.

---

(12) Observe that this condition is fulfilled if for any $z \in G$ there exists an $f \in L^2_h(G)$ such that $f(z) \neq 0$.

(13) Observe that this condition is always true for a bounded domain.
Proof. The construction of functions verifying condition (3.5.15) is done by solving a ∂̄-problem. For more details, the reader may consult [Che-Zha 2002].

From the above theorem we immediately get the following one–dimensional result (see [Che-Zha 2002]).

**Corollary 3.5.3.** Any hyperbolic Riemann surface has a Bergman metric and distance. In particular, any plane domain $D \subset \mathbb{C}$, such that $\mathbb{C} \setminus D$ is not a polar set, has a Bergman distance.

**Remark 3.5.4.** Moreover, any domain $D \subset \mathbb{C}^n$ which carries either a bounded continuous strictly psh function or a negative function $u \in \mathcal{P}SH(D)$ such that $\{z \in D : u(z) < -r\} \subset D$, $r > 0$, satisfies (3.5.15), i.e. $D$ allows a Bergman distance. For more details see [Che-Zha 2002].

Moreover, the following result due to N. Nikolov (private communication) is also a consequence of Theorem 3.5.2.

**Corollary 3.5.5.** Let $D \subset \mathbb{C}^n$ be an unbounded domain. Assume that there are $R > 0$ and $\psi \in \mathcal{P}SH(D \setminus \overline{B(R)})$ such that:

- $\psi < 0$ on $D \setminus \overline{B(R)}$,
- $\lim_{z \to \infty} \psi(z) = 0$,
- $\limsup_{z \to a} \psi(z) < 0$, $a \in (\partial D) \setminus \overline{B(R)}$.

Then $D$ has the Bergman metric.

Proof. Fix a $z_0 \in D$ and choose positive numbers $R_3 > R_2 > R_1 > R$ such that $\|z_0\| < R_1$ and

$$2 \inf_{D \setminus \overline{B(R_2)}} \psi \geq \sup_{D \cap \partial B(R_1)} \psi := c < 0.$$  

Moreover, put

$$d := \inf_{w \in D \cap \partial B(R_1)} \log g_{B(R_1)}(z_0, w) > -\infty,$$

$$u(w) := 2\psi(w)(d/c) - d, \quad w \in D \setminus \overline{B(R)}.$$  

Observe that

$$u(w) \leq d \leq \log g_{B(R_3)}(z_0, w), \quad w \in D \cap \partial B(R_1),$$

$$u(w) \geq 0 \geq \log g_{B(R_3)}(z_0, w), \quad w \in D \cap \partial B(R_2).$$  

Hence, the following function

$$v(w) := \begin{cases} 
\log g_{B(R_3)}(z_0, w), & w \in D \cap B(R_1) \\
\max\{\log g_{B(R_3)}(z_0, w), u(w)\}, & w \in D \cap (B(R_2) \setminus B(R_1)) \\
u(w), & w \in D \cap B(R_2) 
\end{cases}$$

is psh on $D$ with logarithmic pole at $z_0$. Therefore, $v + d \leq \log g_D(z_0, \cdot)$ on $D$. Since $v = u \geq 0$ on $D \setminus B(R_2)$, we have $\log g_D(z_0, \cdot) \geq d$ on $D \setminus B(R_2)$. And if $w \in D \cap B(R_2)$, then $\log g_D(z_0, w) \geq \log g_{B(R_3)}(z_0, w) + d \geq \log g_{B(z_0, R_3 + \|z_0\|)}(z_0, w) + d$.  

Let $\mathcal{B}(z_0, s) \Subset D \cap \mathcal{B}(R_2)$, then there is a $d_1 < 0$ such that $\log g_D(z_0, w) \geq d + d_1$, $w \in D \cap \mathcal{B}(R_2) \setminus \mathcal{B}(z_0, s)$. Therefore, $A_{z_0}(D; d + d_1) \Subset D$ and, since $z_0$ was arbitrarily chosen, Theorem 3.5.2 finishes the proof.

It is an old result due to Bremermann that a bounded $b$–complete domain in $\mathbb{C}^n$ is pseudoconvex. On the other side there is the following sufficient conditions for a bounded domain to be $b$–complete due to Kobayashi.

**Theorem 3.5.6.** Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain.

(a) Assume that

\[
\limsup_{z \to \partial D} \frac{|f(z)|}{\sqrt{K_D(z)}} < \|f\|_{L^2_D(D)}^2, \quad f \in L^2_D(D) \setminus \{0\}.
\]

Then $D$ is $b$–complete.

(a') Let $H \subset L^2_D(D)$ be a dense subspace. Moreover, assume that for any sequence $(z_j)_j \in D$, $z_j \to z_0 \in \partial D$, and any $g \in H$, there is a subsequence $(z_{j_k})_k$ such that

\[
\lim_{k \to \infty} \frac{|g(z_{j_k})|}{\sqrt{K_D(z_{j_k})}} = 0.
\]

Then $D$ is $b$–complete.

(b) For all $z, w \in D$ we have $b_D(z, w) \geq \arccos \frac{|K_D(z, w)|}{\sqrt{K_D(z) K_D(w)}}$.

The statements (a) and (b) are due to Z. Blocki (see [Blo 2002], [Blo 2003]). Observe that (b) explains the connection between the Bergman distance and the Skwarczyński distance (see for more details [J-P 1993]).

**Proof.** (a) Suppose $D$ is not $b$–complete. Then, in virtue of the proof of Lemma 7.6.4 in [J-P 1993], we may find an $f \in L^2_D(D)$, $\|f\|_{L^2_D(D)} = 1$, and real number $\theta_j$ such that

\[
e^{-\varphi_j} \frac{K_D(z_j, z_j)}{K_D(z_j)} \to f \quad \text{in} \quad L^2_D(D).
\]

Therefore, taking the scalar product with $f$, we get $\frac{|f(z_j)|}{\sqrt{K_D(z_j)}} \to \|f\|^2$; a contradiction.

(a') Suppose again that $D$ is not $b$–complete and choose $f$ and $\theta_j$ as above. Moreover, take a $g \in H$ with $\|g - f\|_{L^2_D(D)} < 1/2$. Then, in virtue of our assumption, there is a subsequence $(z_{j_k})_k$ such that

\[
1 \leq k \to \infty \frac{|f(z_{j_k})|}{\sqrt{K_D(z_{j_k})}} \leq \|f\|_{L^2_D(D)} + \frac{|g(z_{j_k})|}{\sqrt{K_D(z_{j_k})}} \to k \to \infty \|g - f\|_{L^2_D(D)} < \frac{1}{2}.
\]

a contradiction.

**Remark 3.5.7.** Most of the results on $b$–completeness will be based on Theorem 3.5.6. In order to verify $b$–completeness one could also try to find good quantitative estimates for the Bergman distance or the Bergman metric near the boundary. For example, there are the following two positive results.
Theorem ([Die-Ohs 1995]). Let \( D \subset \mathbb{C}^n \) be a \( C^2 \)–smooth bounded pseudoconvex domain and let \( z_0 \in D \). Then there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
b_D(z_0, z) \geq C_1 \log |\log(C_2 \text{dist}(z, \partial D))| - 1, \quad z \in D.
\] (3.5.16)

Theorem ([Blo 2002]). Let \( D \) be as above and let \( z_0 \in D \). Then there is a positive constant \( C \) such that
\[
b_D(z_0, z) \geq C \log \frac{1}{\text{dist}(z, \partial D)} \log \log \frac{1}{\text{dist}(z, \partial D)}, \quad z \in D, \text{ sufficiently close to } \partial D.
\] (3.5.17)

In fact, both estimates (3.5.16) and (3.5.17) remain true in a more general situation, namely, for bounded pseudoconvex domains, not necessarily smooth, which allow a good bounded psh exhaustion function.

On the other side, the following example shows that there are certain obstacles, even for smooth domains, to allow a good boundary behavior of the Bergman metric.

Theorem ([Die-Her 2000]). Let \( a \in (0, 1) \). Then there exists a bounded pseudoconvex domain \( D \subset \mathbb{C}^2 \) given as
\[
D = \{ z \in \mathbb{C}^2 : r(z) < 0 \}
\]
with a smooth boundary, \( 0 \in \partial D \), where the defining function \( r \) is of the form \( r(z) = \text{Re} z_1 + b|z_1|^2 + \rho(z_2) \) for suitable \( \rho \in PSH(\mathbb{C}) \), \( \rho(0) = 0 \), and \( b > 0 \) such that there are no positive constant \( C \) and no neighborhood \( U = U(0) \subset \mathbb{C}^2 \) such that
\[
\beta_D(z; X) \geq C \frac{\left| \sum_{j=1}^n \frac{\partial r}{\partial z_j}(z) X_j \right|}{|r(z)| \left( \log \frac{1}{|r(z)|} \right)^{1+a}}, \quad z \in D \cap U.
\]

As a consequence of Theorem 3.5.6 we get (see [Blo-Pfl 1998], [Her 1999])

Theorem 3.5.8. Let \( D \subset \mathbb{C}^n \) be a bounded pseudoconvex domain. Assume that
\[
\lim_{D \ni z \rightarrow \partial D} A_{2n}(A_z(D)) = 0.
\]
Then \( D \) is \( b \)–complete.

In particular, any hyperconvex bounded domain is \( b \)–complete.

Proof. Fix an \( f \in L^2_b(D) \setminus \{0\} \). Using Theorem 3.3.3 we have
\[
\frac{|f(z)|^2}{k_D(z)} \leq C_n \int_{A_z(D)} |f(w)|^2 dA_{2n}(w), \quad z \in D.
\]
Then the assumption and the Theorem 3.5.6(a') immediately gives the proof.

Finally, it suffices to recall that a hyperconvex domain fulfills the condition on the level sets of the Green function. \( \square \)

Remark 3.5.9. In [Che 2003], a similar result is announced even for arbitrary domains, namely:

Let \( D \subset \mathbb{C}^n \) be a (not necessarily bounded) domain. Assume that there is a strictly psh function \( u : D \rightarrow [-1, 0) \) such that all sublevel sets \( \{ z \in D : u(z) < c \}, c \in (-1, 0) \), are relatively compact subsets of \( D \). Then \( D \) is \( b \)–complete.

For weaker results see also [Che-Zha 2002].
A direct consequence of Theorem 3.5.6 is the following sufficient criterion for $b$-completeness.

**Corollary 3.5.10.** Let $D \subset \mathbb{C}^n$ be a bounded $b$-exhaustive domain. Assume that there is a dense subspace $H \subset L^2_b(D)$ such that

\[ \text{any } f \in H \text{ is bounded near } z_0, \quad z_0 \in \partial D. \]

Then $D$ is $b$-complete.

Applying this result together with Theorem 3.1.2 leads to the following sufficient criterion for a plane domain to be $b$-complete (see [Che 2000]).

**Corollary 3.5.11.** Any bounded $b$-exhaustive domain $D \subset \mathbb{C}$ is $b$-complete.

**Remark 3.5.12.** For a bounded pseudoconvex domain, a localization result for the Bergman metric is well known ([J-P 1993]; for a sharper version see also [Her 2003]). This implies that a bounded pseudoconvex domain in $\mathbb{C}^n$ is $b$-complete iff $D$ is locally $b$-complete, i.e., for any $a \in \partial D$ there is an open neighborhood $U = U(a)$ such that any connected component $V$ of $D \cap U$ is $b$-complete.

Due to N. Nikolov [Nik 2003a], there is an analogous result in the plane case for the unbounded situation, namely:

**Theorem 3.5.13.** Let $D \subset \mathbb{C}$ be a domain such that $\mathbb{C} \setminus D$ is not a polar set. Assume that $D$ is locally $b$-complete \(^{(14)}\). Then $D$ is $b$-complete.

The proof of Theorem 3.5.13 is based on the following lemma.

**Lemma 3.5.14.** Let $D \subset \mathbb{C}$ be a domain such that $\mathbb{C} \setminus D$ is not polar. Moreover, let $a \in \partial D$ and $U = U(a)$ be an open neighborhood of $a$. Then there exists a neighborhood $V = V(a) \subset U$ and a constant $C > 0$ such that

\[ C \beta_U(z; 1) \leq \beta_D(z; 1), \quad z \in V \cap D, \]

where $\hat{U}$ denotes that connected component of $D \cap \hat{U}$ with $z \in \hat{U}$.

**Proof.** Since $\mathbb{C} \setminus D$ is not polar, there is an $r_0 > 0$ such that $\mathbb{C} \setminus (D \cup B(a, r_0))$ is not polar. Hence, $\log g_{D \cup B(a, r_0)}$ is harmonic on $(D \cup B(a, r_0)) \setminus \{a\}$. Fix an $r_1 \in (0, r_0)$ and define $D_1 := D \cup B(a, r_1)$. Applying that $g_{D_1}(a, z) \geq g_{D \cup B(a, r_0)}(a, z)$, $z \in D_1$, we have

\[ \inf \{ g_{D_1}(a, z) - |z - a|^2 : z \in \partial B(a, r_1) \cap D \} =: m > -\infty. \]

Put

\[ u(z) := \begin{cases} \max \{ |z - a|^2 + m, \log g_{D_1}(a, z) \}, & \text{if } z \in D \cap B(a, r_1) \\ \log g_{D_1}(a, z), & \text{if } z \in D \setminus B(a, r_1) \end{cases}, \]

Observe that $|z - a|^2 + m \leq g_{D_1}(a, z), z \in D \cap B(a, r_1)$. Therefore, $0 \geq u \in S\!(D_1)$ and $u(z) = |z - a|^2 + m, z \in B(a, r_2)$, for a sufficiently small $r_2 < r_1$.

Choose numbers $0 < r_4 < r_3 < r_2$ and a $C^\infty$ cut-off function $\chi$ such that $\chi \equiv 1$ on $B(a, r_4)$ and $\chi \equiv 0$ outside of $B(a, r_3)$.

\(^{(14)}\) Observe that here the point $\infty$ is counted as a boundary point of $D$; so we assume also that there is a compact set $K \subset \mathbb{C}$ such that any (non empty) connected component of $D \setminus K$ is $b$-complete.
Fix a point $z_0 \in D \cap B(a, r_1)$ and let $\tilde{U}$ be the connected component of $D \cap U$ with $z_0 \in \tilde{U}$. Take an $f \in L^2_b(\tilde{U})$ with $f(z_0) = 0$. Put

$$
\alpha(z) := \begin{cases} 
\overline{\partial}(\chi f)(z), & \text{if } z \in \tilde{U} \\
0, & \text{if } z \in D \setminus \tilde{U}
\end{cases}
$$

Then $\alpha$ is a $\overline{\partial}$-closed $C^\infty_{(0,1)}$-form on $D$ satisfying the following inequality

$$
\int_D |\alpha(z)|^2 e^{-6 \log g_D(z_0, z) - u(z)} dA_2(z) \leq \tilde{C} \int_{\tilde{U}} |f(z)|^2 dA_2(z) < \infty,
$$

where $\tilde{C} > 0$ is independent of $f$ and $z_0$. Observe that the subharmonic weight function is strictly subharmonic near the support of $\alpha$. Therefore, using Hörmander’s $L^2$-theory (15), we get a function $h \in C^\infty(D)$ with $\overline{\partial}h = \alpha$ on $D$ such that

$$
\|h\|_{L^2(D)}^2 \leq \int_D |h(z)|^2 e^{-6 \log g_D(z_0, z) - u(z)} dA_2(z) \leq C' \|f\|^2_{L^2(\tilde{U})},
$$

where $C'$ is a positive number which is independent of $f$ and $z_0$. Moreover, since the second integral is finite, it follows that $h(z_0) = h'(z_0) = 0$. Hence, the function

$$
\hat{f}(z) := \begin{cases} 
(\chi f)(z) - h(z), & \text{if } z \in \tilde{U} \\
-h(z), & \text{if } z \in D \setminus \tilde{U}
\end{cases}
$$

is holomorphic on $D$ satisfying $\hat{f}(z_0) = 0$, $\hat{f}'(z_0) = f'(z_0)$, and $\|\hat{f}\|_{L^2(D)} \leq C \|f\|_{L^2(\tilde{U})}$, where $C > 0$ is independent of $f$ and $z_0$. Therefore, in virtue of (3.5.14), we get $\tilde{C} \beta_D(z_0; 1) \geq \beta_{\tilde{U}}(z_0; 1)$. Since $z_0$ was arbitrary, the lemma is proved.

Now, we turn to the proof of Theorem 3.5.13.

**Proof of Theorem 3.5.13.** First of all, let us mention that, in virtue of Corollary 3.5.3, $D$ has a Bergman metric.

Suppose now that $D$ is not $b$-complete. Then there is a $b$-Cauchy sequence $(z_j)_j \subset D$ with $z_j \rightarrow a \in \partial D$ or $z_j \rightarrow \infty$. The second case can be reduced to the first one using the biholomorphic transformation $z \mapsto \frac{1}{z - c}$, where $c \notin D$. So we only have to deal with the first case.

By the assumption, there is an open neighborhood $U = U(a)$ such that any connected component of $U \cap D$ is $b$-complete. Fix a positive $r_1$ such that $\mathbb{B}(a, r_1) \subset U$. Applying Lemma 3.5.14, we may find positive numbers $r_2 < r_1$ and $C$ such that and $C \beta_D(z; 1) \leq \min(\beta_D(z_1), \beta_{\tilde{U}}(z_1))$, $z \in D \cap \mathbb{B}(a, r_2)$, where $\tilde{U}$ and $\tilde{U}$ denote the connected components of $D \cap U$ and $D \cap \mathbb{B}(a, r_1)$, respectively, with $z \in \tilde{U} \cap \tilde{U}$. Choose an

---

(15) Here we use the following form of Hörmander’s result.

**Theorem.** Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain, $\varphi \in PSH(D)$, and $\alpha \in C^\infty_{(0,1)}(D)$. Assume that $\overline{\partial} \alpha = 0$ and that on an open set $U \subset D$, sup $\alpha \subset U$, the function $\varphi$ can be written as $\varphi = \psi + \chi$, $\psi, \chi \in PSH(U)$, such that $L^2_w(z; X) \geq C \|X\|^2$, $z \in U$, $X \in \mathbb{C}^n$. Then there exists an $h \in C^\infty(D)$, $\overline{\partial} h = \alpha$, such that $\int_D |h|^2 e^{-\varphi} dA_2 \leq C' \int_D |\alpha|^2 e^{-\varphi} dA_2$, where $C' > 0$ depends only on $C$. 
\[ r_3 \in (0, r_2). \] Put

\[ d := \inf \{ b_{D}(z, w) : z \in \partial B(a, r_3) \cap D, \ w \in \partial B(a, r_2) \cap D, \ z, w \in \tilde{U}, \ \tilde{U} \] a connected component of \( D \cap B(a, r_3) \}. \]

In virtue of the inequality \( c \leq b \), it follows that \( d > 0 \). So we may take an index \( k_0 \in \mathbb{N} \) such that \( b_D(z_k, z_\ell) < \frac{cd}{2} \) and \( z_k \in B(a, r_3) \), \( k, \ell \geq k_0 \).

Fix such \( k, \ell \) with \( z_k \neq z_\ell \). Then there is a \( C^1 \)-curve \( \alpha_{k,\ell} : [0, 1] \to D \) such that

\[ 2b_D(z_k, z_\ell) > \int_0^1 \beta_D(\alpha_{k,\ell}(t); \alpha_{k,\ell}'(t))dt. \]

Suppose this curve is not lying in \( B(a, r_1) \). Then there are numbers \( 0 < s_1 < s_2 < 1 \) such that \( \alpha_{k,\ell}(s_1) \in \partial B(a, r_3) \), \( \alpha_{k,\ell}(s_2) \in \partial B(a, r_2) \), and \( \alpha_{k,\ell}([s_1, s_2]) \subset B(a, r_1) \). Hence,

\[ 2b_D(z_k, z_\ell) > \int_{s_1}^{s_2} \beta_D(\alpha_{k,\ell}(t); \alpha_{k,\ell}'(t))dt \geq Cb_D(z_k, z_\ell) \geq dc, \]

where \( \tilde{U} \) is the connected component of \( D \cap B(a, r_1) \) containing this part of the curve; a contradiction.

Hence, we obtain for \( k, \ell \geq k_0 \) that \( Cb_{\tilde{U}_{k,\ell}}(z_k, z_\ell) \leq b_D(z_k, z_\ell) \), where \( \tilde{U}_{k,\ell} \) denotes the connected component of \( U \cap D \) which contains the curve \( \alpha_{k,\ell} \). Hence, \( (z_j)_j \) is even a \( b_{\tilde{U}_{k,\ell}} \)-Cauchy sequence; a contradiction. \( \square \)

**Remark 3.5.15.** Let \( D \subset \mathbb{C} \) be an unbounded \( b \)-complete domain. Due to N. Nikolov (private communication) the following inverse statement to that of Theorem 3.5.13 is true: For any open disc \( U \subset \mathbb{C} \), any connected component of \( U \cap D \) (resp. \( D \setminus \bar{U} \)) is also \( b \)-complete.

Moreover, there is the following general result for balanced domains.

**Theorem 3.5.16.** Let \( D \subset \mathbb{C}^n \) be a bounded pseudoconvex balanced domain. Then \( D \) is \( b \)-complete.

**Proof.** Recall that any \( f \in \mathcal{O}(D) \) can be written as a series \( \sum_{k=1}^\infty Q_k \), where \( Q_k \) are homogeneous polynomials, and that this convergence is an \( L^2_\beta \)-convergence. Therefore, the bounded holomorphic functions on \( D \) are dense in \( L^2_\beta (D) \). Then Theorem 3.3.9 and Corollary 3.5.10 finishes the proof. \( \square \)

[?] Characterize the \( b \)-complete bounded circular pseudoconvex domains [?]

**Remark 3.5.17.** There are also sufficient conditions for Hartogs domains with \( m \)–dimensional fibers to be \( b \)-complete (see [Jar-Pfl-Zwo 2000]).

**Theorem.** Let \( D \subset \mathbb{C}^n \) be a domain and let \( G_D \) be bounded pseudoconvex Hartogs domain with \( m \)-dimensional balanced fibers.

(a) Assume that \( D \) is \( b \)-exhaustive, that \( H^\infty (D) \) is dense in \( L^2_\beta (D) \), and that there is an \( \varepsilon > 0 \) such that \( D \times P(0, \varepsilon) \subset G_D \).

Then \( G_D \) is \( b \)-complete.

(b) Assume that \( D \) is \( c^\epsilon \)-complete, then \( G_D \) is \( b \)-complete.
For a proof see [Jar-Pfl-Zwo 2000]. For further results on $b$–complete Hartogs domains see also [Che 2001b]. Is there a complete characterization of such domains which are $b$–complete?

Moreover, the following result may be found in [Che 2001a].

**Theorem.** Let $u \in PSH(C^n)$, $u \not\equiv -\infty$, and $h \in O(C^n)$, $h \not\equiv 0$, be such that $u \in C(C^n \setminus u^{-1}(-\infty))$. Let $r > 0$ and assume that

$$
\Omega := \{(z', z'') \in B_n(r) \times B_m(r) \subset C^n \times C^m : u(z') + e^{\frac{1}{2r^2 \log r_k}} < 1\}
$$

is a domain. Then $\Omega$ is $b$–complete.

If, in addition, there is a point $(z'_0, z''_0) \in B_n(r) \times B_m(r)$ with $u(z'_0) = -\infty$, $h(z''_0) = 0$, then $\Omega$ is not hyperconvex.

Observe that the boundary behavior of the level sets of the Green function implies both $b$–exhaustiveness and $b$–completeness. We already saw that there exist $b$–exhaustive domains not being $b$–complete. It was a long standing question whether any $b$–complete domain is automatically $b$–exhaustive. The first counterexample to that question was given by W. Zwonek [Zwo 2001a] (see also [Zwo 2002]). The following Theorem 3.5.18 (see [Juc 2003]) gives even a large variety of domains that are $b$–complete but not $b$–exhaustive.

**Theorem 3.5.18.** Let $D \subset C$ be a Zalcman domain as in Corollary 3.3.22. Then:

$$
D \text{ is } b\text{-complete } \iff \sum_{k=1}^{\infty} \frac{1}{r_k \sqrt{- \log r_k}} = \infty.
$$

**Proof.** The proof “$\Rightarrow$” is similar to the one of Theorem 3.3.19; so it is omitted here.

Proof of “$\Leftarrow$”: Suppose that $D$ is not $b$–complete. Then $D$ is not $b$–exhaustive. Therefore, in view of Corollary 3.3.22, we have

$$
\sum_{k=1}^{\infty} \frac{1}{r_k \sqrt{- \log r_k}} = \infty \quad \text{and} \quad \lim_{j \to \infty} \frac{1}{- x_j^2 \log r_j} = 0. \tag{3.5.18}
$$

Moreover, there is a $b_D$–Cauchy–sequence $(z_k)_k \subset D$ with $\lim_{k \to \infty} z_k = 0$. We may even assume that $b_D(z_k, z_{k+1}) < \frac{1}{2^k}$. So there exist $C^1$–curves $\gamma_k : [0, 1] \to D$ such that $L_{\beta_D}(\gamma_k) < 1/2^k$. Gluing all these curves together we obtain a piecewise $C^1$–curve $\gamma : [0, 1] \to D$ with a finite $\beta_D$–length.

We claim that the Bergman kernel remains bounded along $\gamma$. In fact, if not then there is a sequence $(w_k)_k \subset \gamma([0, 1))$ such that

$$
\lim_{k \to \infty} k_D(w_k) = \infty, \quad \lim_{k \to \infty} w_k = 0.
$$

Obviously, the sequence $(w_k)_k$ is again a $b_D$–Cauchy–sequence. As in the proof of Theorem 3.5.6, there exist an $f \in \mathcal{L}^2(D)$ and a subsequence $(w_{k_j})_j$ such that

$$
\lim_{j \to \infty} \frac{|f(w_{k_j})|^2}{k_D(w_{k_j})} = 1.
$$
Applying Theorem 3.1.2, we find a \( g \in L^2(D) \), locally bounded near 0, such that \( \|f - g\|_{L^2(D)} < 1/2 \). Therefore,

\[
0 \overset{j \to \infty}{\longrightarrow} \frac{|g(w_k)|}{\sqrt{k_D(w_k)}} \geq \frac{|f(w_k)|}{\sqrt{k_D(w_k)}} - \|f - g\|_{L^2(D)} \geq \frac{|f(w_k)|}{\sqrt{k_D(w_k)}} - \frac{1}{2} \overset{j \to \infty}{\longrightarrow} \frac{1}{2};
\]

a contradiction. Hence, there is a positive \( C \) such that \( k_D(\gamma(t)) \leq C, \ t \in [0, 1) \).

To be able to continue we need the following lemma.

**Lemma 3.5.19.** Let \( D \) be a domain as above satisfying (3.5.18) and let \( \gamma : [0, 1) \to D \) be a piecewise \( C^1 \)-curve with \( \lim_{t \to 1} \gamma(t) = 0 \). Then

\[
\lim_{\tau \to 1} \int_0^\tau \sqrt{M_D(\gamma(t); \gamma'(t))} dt = \infty.
\]

**Proof.** We may assume that \( |\gamma(0)| > x_1 \) and that \( x_1 \sqrt{-\log r_1} < x_j \sqrt{-\log r_j}, \ j \geq j_0 \) for a suitable \( j_0 \) (use (3.5.18)). Now, fix an \( N \in \mathbb{N} \), \( N \geq j_0 \), and let \( z_N \in D \) be an arbitrary point with \( x_{N+2} \leq |z_N| \leq x_{N+1} \).

We define

\[
f := f_{\mathcal{E}(x_1, r_1)} - \frac{x_N - z_N}{x_1 - z_N} f_{\mathcal{E}(x_N, r_N)},
\]

where \( f_K \) denotes the Cauchy transform of \( K \). Or more explicit, we have

\[
f(z) = \frac{1}{x_1 - z} - \frac{x_N - z_N}{x_1 - z_N} \frac{1}{x_N - z}, \ \ z \in D.
\]

Therefore, we see that \( f(z_N) = 0 \) and \( f'(z_N) = \frac{2}{(x_1 - z_N)^2(x_N - z_N)} \). What remains is to estimate the \( L^2(D) \)-norm of the function \( f \). Applying the relation between \( x_n \) and \( x_{n+1} \), we get

\[
\|f\|_{L^2(D)} \leq \left\| f_{\mathcal{E}(x_1, r_1)} \right\|_{L^2(D)} + \frac{|x_N - z_N|}{|x_1 - z_N|} \left\| f_{\mathcal{E}(x_N, r_N)} \right\|_{L^2(D)} \leq C_2 \frac{|x_N - z_N|}{|x_1 - z_N|} \sqrt{-\log r_N},
\]

where \( C_1, C_2 \) are positive constants, independent of \( N \) and \( z_N \).

Therefore, if \( x_{N+2} \leq |z| \leq x_{N+1} \) then:

\[
\sqrt{M_D(z; X)} \geq |X| \left| \frac{x_1 - x_N}{x_1 - \gamma(t)} \right| \left| \frac{x_N - x_N}{x_1 - \gamma(t)} \right| \frac{1}{x_1 \sqrt{-\log r_N}} \geq |X| \frac{C_3}{x_1 \sqrt{-\log r_N}},
\]

where \( C_3 > 0 \) is a constant (use again that \( x_{k+1} \leq \Theta_2 x_k \) for all \( k \)). Finally, we obtain

\[
\lim_{\tau \to 1} \int_0^\tau \sqrt{M_D(\gamma(t); \gamma'(t))} dt \geq \sum_{N=0}^{\infty} C_3 \frac{x_{N+1} - x_N}{x_N \sqrt{-\log r_N}} \geq C_4 \sum_{N=0}^{\infty} \frac{1}{x_N \sqrt{-\log r_N}} = \infty,
\]

where \( C_4 > 0 \). Hence, the proof of this lemma is complete. \( \square \)

Now, applying Lemma 3.5.19 leads to the following contradiction:

\[
\infty > \lim_{\tau \to 1} \int_0^\tau \beta_D(\gamma(t); \gamma'(t)) dt \geq \frac{1}{C} \int_0^\tau \sqrt{M_D(\gamma(t); \gamma'(t))} dt = \infty.
\]

\( \square \)
The boundary behavior of the Bergman metric on a Zalcman domain is partially described in the following result whose proof is based on methods of the proof of Theorem 3.5.18.

**Theorem 3.5.20** ([Juc 2003]). Let $D$ be a domain as in Theorem 3.5.18.

(a) If $\sum_{k=1}^{\infty} \frac{1}{x_k^2 - \log x_k} < \infty$, then $\limsup_{(1,0) \neq t \to 0} \beta_D(t;1) < \infty$.

(b) If $\limsup_{(1,0) \neq t \to 0} \beta_D(t;1) < \infty$, then $\limsup_{k \to \infty} \frac{1}{x_k^2 - \log x_k} < \infty$.

It seems to be open how to characterize those Zalcman domains that are $\beta$-exhaustive, i.e. $\lim_{z \to \partial D} \beta_D(z;1) = \infty$.

The $b$-completeness means heuristically that boundary points are infinitely far away from inner points. So one might think that for a $b$-complete domain the Bergman metric $\beta_D$ becomes infinite at the boundary. The following example shows that this is not true.

**Example 3.5.21.** There exists a $b$-complete bounded domain $D$ in the plane which is not $\beta_D$-exhaustive, i.e. there is a boundary sequence $(w_k)_k \subset D$ such that $(\beta_D(w_k;1))_{k \in \mathbb{N}}$ is bounded ([Pfl-Zwo 2003a]). To be more precise:

Put

$$x_n := \frac{1}{2n+1} + \frac{1}{2n+2}, \quad z_{n,j} := \exp\left(\frac{j \pi n}{2^{4n}}\right), \quad n \in \mathbb{N}, \quad j = 0, \ldots, 2^{4n} - 1.$$  

Moreover, let $r_n := \exp(-C_1 2^{4n})$, $n \in \mathbb{N}$, where $C_1 > 0$ is chosen such that

- the discs $B(z_{n,j}, r_n) \subset \mathbb{C}$, $n \in \mathbb{N}$, $j = 0, \ldots, 2^{4n} - 1$, are pairwise disjoint,
- $B(0, r_n) \subset A_\beta(0)$, $n \in \mathbb{N}$, $j = 0, \ldots, 2^{4n} - 1$.

Then there is a sequence $(n_k)_k \subset \mathbb{N}$ such that the domain

$$D := E \setminus \left( \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{2^{4n_k} - 1} B(z_{n_k,j}, r_n) \right)$$

is a domain satisfying the above desired properties.

3.5.1. **Reinhardt domains and $b$-completeness.** In the class of pseudoconvex Reinhardt domains there is a complete geometric characterization of $b$-complete domains (see [Zwo 1999b], [Zwo 2000b]).

Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then $\Omega := \Omega_D := \partial D$ is a convex domain in $\mathbb{R}^n$. Let us fix a point $a \in \Omega$. Put

$$\mathcal{E}(\Omega, a) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \Omega\}.$$  

It is easy to see that $\mathcal{E}(\Omega, a)$ is a closed convex cone with vertex at 0, i.e. $tx \in \mathcal{E}(\Omega, a)$ for all $x \in \mathcal{E}(\Omega, a)$ and $t \in \mathbb{R}_+$. Moreover, this cone is independent of the point $a$, i.e. $\mathcal{E}(\Omega, a) = \mathcal{E}(\Omega, b)$, $b \in \partial \Omega$. So we will write shortly $\mathcal{E}(\Omega) := \mathcal{E}(\Omega, a)$. Observe that $\mathcal{E}(\Omega) = \{0\}$ iff $\Omega \subset \subset \mathbb{R}^n$.

We define now

$$\tilde{\mathcal{E}}(D) := \{v \in \mathcal{E}(\Omega_D) : \exp(a + \mathbb{R}_+ v) \subset D\}, \quad \mathcal{E}'(D) := \mathcal{E}(\Omega_D) \setminus \tilde{\mathcal{E}}(D).$$  

Observe that the definition of $\tilde{\mathcal{E}}(D)$ and $\mathcal{E}'(D)$ is independent of the point $a$. 
With the help of this geometric notions, there is the following complete description of those bounded Reinhardt domains which are Bergman complete.

**Theorem 3.5.22** ([Zwo 1999b]). Let \( D \subset \mathbb{C}^n \) be a bounded pseudoconvex Reinhardt domain. Then the following conditions are equivalent:

(i) \( D \) is \( b \)-complete;

(ii) \( \mathcal{C}(D) \cap Q^n = \emptyset \).

**Example 3.5.23.** Put

\[
D_1 := \{ z \in \mathbb{C}^2 : z_1^2/2 < |z_2| < 2|z_1|^2, |z_1| < 2 \}.
\]

Obviously, \( D_1 \) is a bounded pseudoconvex domain which contains the point \((1,1)\). Then it turns out that \( \mathcal{C}(D_1) = \mathbb{R}_{>0}(-1,-2) \); so it contains the rational vector \((-1,-2)\).

Using the map

\[
\Phi : \mathbb{C}^2_+ \rightarrow \mathbb{C}^2_+, \quad \Phi(z) := (z_1^{3/2}, z_2^{1/2}), \quad z = (z_1, z_2),
\]

we see that \( D_1 \) is biholomorphic to

\[
\tilde{D}_1 := \{ z \in \mathbb{C}^2_+ : 1/2 < |z_2| < 2, |z_1z_2| < 2 \}.
\]

It may be directly seen that \( \tilde{D}_1 \) and, therefore also \( D_1 \), is not \( b \)-complete.

On the other hand, let

\[
D_2 := \{ z \in \mathbb{C}^2 : z_2 |z_1|^{1/2} < |z_2| < 2|z_1|^2, |z_1| < 2 \}.
\]

Again, \( D_2 \) is a bounded pseudoconvex Reinhardt domain; now a simple calculation gives \( \mathcal{C}(D_2) = \mathbb{R}_{>0}(-1,-\sqrt{2}) \); i.e. \( \mathcal{C}(D_2) \) does not contain any rational vector. Hence the Theorem 3.5.22 tells us the \( D_2 \) is \( b \)-complete. Recall that \( D_2 \) is not hyperconvex.

The next example can be found in [Her 1999]. Let

\[
D := \{ z \in \mathbb{C}^2 : |z_2|^2 < \exp(-1/|z_1|^2), |z_1| < 1 \}.
\]

Again, \( D \) is a bounded pseudoconvex Reinhardt domain. Here we have \( \mathcal{C}(D) = \tilde{\mathcal{C}}(D) = \{0\} \times \mathbb{R}_+ \) and \( \mathcal{C}(D) = \emptyset \). So Theorem 3.5.22 gives that \( D \) is \( b \)-complete (in [Her 1999], a direct proof of this fact is presented). Again, observe that \( D \) is not hyperconvex.

For the proof of Theorem 3.5.22 we need the following lemma.

**Lemma 3.5.24.** Let \( C \subset \mathbb{R}^n \) be a convex closed cone with \( C \cap Q^n = \{0\} \). Assume that \( C \) contains no straight lines. Then for any positive \( \delta \) and any vector \( v \in C \setminus \{0\} \) there is a \( \beta \in \mathbb{Z}^n \) such that

\[
\langle \beta, v \rangle > 0 \quad \text{and} \quad \langle \beta, w \rangle < \delta, \; w \in C, \ |w| = 1.
\]

Since this lemma is based on the geometric number theory, we will omit its proof. For more details, we refer to [Zwo 1999b].

**Proof of Theorem 3.5.22.** In a first step we are going to verify (i) \(\Rightarrow\) (ii):

Suppose that (ii) does not hold, i.e. there is a non trivial vector \( v \in \mathcal{C}(D) \cap Q^n \). We may assume that \( 0 \in \log D, \; v = (v_1, \ldots, v_n) \in \mathbb{Z}^n_+ \), and that \( v_1, \ldots, v_n \) are relatively prime. It suffices to see that the Bergman length \( L_{\beta_0} \) of the curve \((0,1) \overset{\gamma}{\rightarrow} (t^{v_1}, \ldots, t^{v_n}) \in D \) is finite.
In fact, put \( \varphi(\lambda) := (\lambda^{-v_1}, \ldots, \lambda^{-v_n}) \), \( \lambda \in E_* \). Then \( \varphi \in \mathcal{O}(E_*, D) \). Now let \( u(\lambda) := k_D(\varphi(\lambda)) \), \( \lambda \in E_* \). To continue we need a part of the following lemma (see [Zwo 2000b]).

**Lemma 3.5.25.** Let \( D \subset \mathbb{C}^n \) be a pseudoconvex Reinhardt domain, \( \alpha \in \mathbb{Z}^n \), and \( p \in (0, \infty) \). Then the following properties hold:

- the monomial \( z^\alpha \) belongs to \( L^p(D) \) iff \( \langle \frac{\partial}{\partial \alpha + 1}, v \rangle < 0 \) for any \( v \in \mathcal{E}(D) \) \( \setminus \{0\} \);
- if \( \langle \alpha, v \rangle < 0 \) for any \( v \in \mathcal{E}(D) \) \( \setminus \{0\} \), then \( z^\alpha \in \mathcal{H}^\infty(D) \);
- if \( z^\alpha \in \mathcal{H}^\infty(D) \), then \( \langle \alpha, v \rangle \leq 0 \) for any \( v \in \mathcal{E}(D) \).

In virtue of Lemma 3.5.25 \((p = 2)\) it follows that

\[
\begin{align*}
  u(\lambda) &= \sum_{\alpha \in \mathbb{Z}^n: (\alpha + 1, v) < 0} a_\alpha |\lambda|^{-2\langle \alpha, v \rangle} = \sum_{j = j_0}^{\infty} b_j |\lambda|^{2j},
\end{align*}
\]

where \( b_{j_0} \neq 0 \). Therefore,

\[
\beta_D(\varphi(\lambda); \varphi'(\lambda)) = \frac{\partial^2 \log u(\lambda)}{\partial \lambda \partial \lambda^*} = \frac{\partial^2}{\partial \lambda \partial \lambda^*} \left( \log \sum_{j = j_0}^{\infty} b_j |\lambda|^{2j-j_0} \right) \quad (16).
\]

Obviously, the last expression remains bounded along \((0, 1)\), which gives the desired contradiction.

Now we turn to the proof of \((ii) \implies (i)\):

We start with the following observation: put \( \mathcal{E} := \text{span}\{z^\alpha : z^\alpha \in L^2(D)\} \). Then \( \mathcal{E} \) is a dense subspace of \( L^2(D) \). In virtue of Theorem 3.5.6, we have only to show that for any point \( z^0 \in \partial D \) and for any sequence \((z_j^k) \subset D\) with \( \lim_{j \to \infty} z^j = z_0 \) we find a subsequence \((z^{j_k})_k\) such that

\[
\frac{|f(z^{j_k})|}{\sqrt{k_D(z^{j_k})}} \to 0, \quad f \in \mathcal{E}. \quad (3.5.19)
\]

First, we discuss the case when \( z^0 \) satisfies the following property: if \( z^0_j = 0 \) then \( V_j \cap D \neq \emptyset, j = 1, \ldots, n \), where \( V_j := \{z \in \mathbb{C}^n : z_j = 0\} \).

Fix an \( \alpha \in \mathbb{Z}^n \) such that \( z^\alpha \in L^2(D) \). Then \( \alpha_j \geq 0 \) for all \( j \) with \( z^0_j = 0 \). So it suffices to verify that \( k_D(z) \to \infty \) \( z \to z^0 \). Without loss of generality, we may assume that \( z^0_j = 0, j = 1, \ldots, s, \) and \( z^0_j \neq 0, j = s + 1, \ldots, n \). Obviously, \( s < n \). Then there is an \( R > 0 \) such that \( D \subset B_s(0, R) \times \mathbb{C}^{n-s} \), where \( \mathbb{C}^{n-s} \) denotes the projection of \( \mathbb{C}^n \) onto \( \mathbb{C}^{n-s} \) if \( s \geq 1 \) or the identity if \( s = 0 \). Then \( \bar{\pi}_s(D) \) is a bounded pseudoconvex Reinhardt domain with \( \bar{\pi}_s(z^0) \in \partial \bar{\pi}_s(D) \), where all coordinates of \( \bar{\pi}_s(z^0) \) are different from zero. Hence, \( \bar{\pi}_s(D) \) satisfies the general outer cone condition at \( \bar{\pi}_s(z^0) \). In virtue of Theorem 6.1.17 in [J-P 1993], it follows that \( \lim_{z \to z^0 \in \mathbb{C}^n} k_{\bar{\pi}_s(D)}(z^0) = \infty \). Using the monotonicity and the product formula of the Bergman kernel, we finally get

\[
k_D(z) \geq k_{B_s(0, R)}(z')k_{\bar{\pi}_s(D)}(z^0) \to \infty \quad (D \ni \bar{z} \to z^0 \in \mathbb{C}^n)
\]

In the remaining part of the proof we assume that there is at least one \( j \) such that \( z^0_j = 0 \), but \( V_j \cap D = \emptyset \).

\(^{(16)}\) Observe here that \( \log |\lambda|^{2j_0} \) is harmonic on \( E_* \).
Hence, the assumption of Theorem 3.5.6 is fulfilled.

Let \( v \in (Q^n \cap \mathcal{C}(D)) \setminus \{0\} \). Then, by assumption, we know that \( v \in \mathcal{E}(D) \). Therefore, \( \lim_{t \to \infty} \exp(tv) = w \in D \). So, if \( v_j < 0 \) then \( w_j = 0 \), and if \( v_j = 0 \) then \( w_j = 1 \). In particular, if there is a \( v \in \mathcal{C}(D) \cap Q^n \), \( v_j < 0 \), then \( j \leq k \).

Observe that \( \mathbb{R}^k \times \{0\}^{n-k} \subset \mathcal{C}(D) \). Now, we claim that for any \( v \in \mathcal{C}(D) \setminus (\mathbb{R}^k \times \{0\}^{n-k}) \) we have that \( v \notin \mathbb{R}^k \times \mathbb{Q}^{n-k} \).

In fact, suppose that \( v \in \mathbb{R}^k \times \mathbb{Q}^{n-k} \). Then \( v_j < 0 \) for some \( j > k \). Then we may choose a suitable vector \( w \in \mathbb{R}^k \times \{0\}^{n-k} \subset \mathcal{C}(D) \) such that \( \tilde{w} := v + w \in \mathcal{C}(D) \cap \mathbb{Q}^n \) and \( \tilde{w} < 0 \). Hence, \( j \leq k \); a contradiction.

Put \( \pi : \mathbb{R}^n \to \mathbb{R}^n \), \( \pi(x) := (0, \ldots, 0, x_{k+1}, \ldots, x_n) \), where \( x = (x_1, \ldots, x_n) \). Then \( \pi(\mathcal{C}(D)) \) is a closed convex cone in \( \{0\}^k \times \mathbb{R}^n \). In virtue of the above property, we conclude that

\[
\pi(\mathcal{C}(D)) \cap (\{0\}^k \times \mathbb{Q}^{n-k}) = \{0\}.
\]

Recall that \( z_{k+1}^0 = 0 \). Now, let \( z^j \in D \cap \mathcal{C}_*^n \) be a sequence tending to \( z^0 \) \((18)\). Put \( x^j := (\log |z_1^j|, \ldots, \log |z_n^j|) \in \mathbb{R}^n \). Obviously, \( \|x^j\| \to \infty \). Moreover, without loss of generality, we may assume that the sequence \( (x^j/\|x^j\|)_j \) converges to a vector \( \tilde{v} \in \mathcal{C}(D) \).

Fix an \( \alpha \in \mathbb{Z}^n \) such that \( z^\alpha \in L^2_k(D) \). Then, using Lemma 3.5.25, we conclude that

\[
\inf\{-\langle \alpha + 1, w \rangle : w \in \mathcal{C}(D), \|w\| = 1\} =: \delta_0 > 0.
\]

Two cases have to be discussed.

Case 1: \( \tilde{v}_j < 0 \) for some \( j > k \).

Applying Lemma 3.5.24 for \( C = \pi(\mathcal{C}(D)) \), \( v = \pi(\tilde{v}) \), and \( \delta_0 \), we get the existence of a \( \beta \in \{0\}^k \times \mathbb{Z}^{n-k} \) such that

\[
\langle \beta, \tilde{v} \rangle = \langle \beta, \pi(\tilde{v}) \rangle > 0, \quad \langle \beta, w \rangle = \|\pi(w)\| \langle \beta, \pi(w) \rangle < \delta, w \in \mathcal{C}(D), \pi(w) \neq 0.
\]

Observe that \( \langle \beta, w \rangle = 0 \) if \( \pi(w) = 0 \).

Then \( z^{\alpha + \beta} \in L^2_k(D) \) (use Lemma 3.5.25) and

\[
\frac{|z_j^\alpha|}{\sqrt{K_D(z^j)}} \leq \|z^{\alpha + \beta}\|_{L^2_k(D)} \|z_j^\alpha\|_{L^2_k(D)} = \|z^{\alpha + \beta}\|_{L^2_k(D)} \|z^j\|_{\mathbb{R}^n} - \|z^\beta\|_{\mathbb{R}^n} \to 0, \quad j \to \infty.
\]

Hence, the assumption of Theorem 3.5.6 is fulfilled.

Case 2: \( \tilde{v}_{k+1} = \cdots = \tilde{v}_n = 0 \).

Recall that \( \|\pi(x^j)\| \to \infty \). So we may assume that

\[
\pi(x^j) \to \tilde{w} = (0, \ldots, 0, \tilde{w}_{k+1}, \ldots, \tilde{w}_n).
\]

If \( \tilde{w} \in \pi(\mathcal{C}(D)) \), then, in virtue of Lemma 3.5.24, there is a \( \beta \in \{0\}^k \times \mathbb{Z}^{n-k} \) such that \( \langle \beta, \tilde{w} \rangle > 0 \) and \( \langle \beta, w \rangle < \delta_0, w \in \mathcal{C}(D) \setminus \{0\} \).

(17) Then necessarily, \( k < n \).

(18) Observe that it suffices to prove (3.5.19) for sequences in \( \mathcal{C}_*^n \).
If \( \tilde{w} \notin \pi(\mathcal{C}(D)) \), let \( \tilde{C} \) be the smallest convex closed cone containing \( \pi(\mathcal{C}(D)) \) and \( -\tilde{w} \). Then \( \tilde{C} \subset \{0\}^k \times \mathbb{R}^{n-k} \) and \( \tilde{w} \notin \tilde{C} \). Therefore,
\[
\{ \tilde{\beta} \in \{0\}^k \times \mathbb{R}^{n-k} : (\tilde{\beta}, u) < 0, u \in \tilde{C} \setminus \{0\} \}
\]
is a non-empty convex open cone (see [Vla 1993], §25). So it contains a \( \beta \in \{0\}^k \times \mathbb{Z}^{n-k} \).

Thus, \( (\tilde{\beta}, -\tilde{w}) < 0 \) and \( (\beta, w) = \|\pi(w)\|((\beta, \pi(w)) < 0 < \delta_0, w \in \mathcal{C}(D), \pi(w) \neq 0 \).

Now we are able to complete the proof as in case 1 using the \( \beta \) we just constructed.

Namely, we conclude that \( z^{\alpha+\beta} \in L^2_h(D) \) and
\[
\frac{|(z^j)^\alpha|}{\sqrt{k_D(z^j)}} \leq \|z^{\alpha+\beta}\|_{L^2_h(D)}|z^{\frac{1}{2} - \beta}| = \|z^{\alpha+\beta}\|_{L^2_h(D)}|z^{\frac{1}{2} - \beta}| \xrightarrow{j \to \infty} 0.
\]

Hence, Theorem 3.5.6 may be applied. \( \square \)

Finally, we will prove that part of Lemma 3.5.25 used during the proof of Theorem 3.5.22.

**Proof of Lemma 3.5.25.** We restrict ourselves to prove only the following statement (the other ones in Lemma 3.5.25 may be taken as an exercise!):

if \( D \) is as in Theorem 3.5.22 (in particular, \( D \) is bounded)
and if \( \langle \alpha + 1, v \rangle < 0, \ v \in \mathcal{C}(D) \setminus \{0\} \), then \( z^\alpha \in L^2_h(D) \).

Assume that \( \mathcal{C}(D) \neq \{0\} \). Then there is a \( \delta_0 < 0 \) such that \( \langle \alpha + 1, v \rangle < \delta_0, v \in \mathcal{C}(D), \|v\| = 1 \).

We claim that for any \( \varepsilon > 0 \) there is a cone \( T \) such that \( \log D \setminus T \) is bounded and \( \|w - v\| < \varepsilon, v \in \mathcal{C}(D), \|v\| = 1 \).

Indeed, fix an \( \varepsilon > 0 \) and let \( h \) be the Minkowski function of the convex set \( \log D \).

Observe that \( h \) is continuous and \( h^{-1}(0) = \mathcal{C}(D) \). Therefore, there is a \( \delta > 0 \) such that
\[
\{ w \in \mathbb{R}^n : h(w) \leq \delta, \|w\| = 1 \} \subset \{ w \in \mathbb{R}^n : \|w\| = 1, \exists v \in \mathcal{C}(D) : \|v\| = 1, \|v - w\| < \varepsilon \}.
\]

Set \( T \) as the smallest cone containing \( \{ w \in \mathbb{R}^n : h(w) \leq \delta, \|w\| = 1 \} \). Then \( \log D \setminus T \) is bounded; otherwise there would exist an unbounded sequence \( x^j \in \log D \setminus T \) such that \( h(x^j) < 1 \). Therefore, \( h(x^j) = \frac{1}{\|x^j\|}, \) i.e. \( x^j \in T \) for large \( j \); a contradiction.

Now observe that \( \langle \alpha + 1, v \rangle \leq \frac{\varepsilon_0}{2\|v\|}, v \in T \), and
\[
\int_D |z^{\alpha}|^2 dA_n(z) < \infty \iff \int_{\log D} e^{2(\alpha + 1,x)} dA_n(x) < \infty \iff \int_T e^{2(\alpha + 1,x)} dA_n(x) < \infty.
\]

So it remains to estimate the last integral. We get
\[
\int_T e^{2(\alpha + 1,x)} dA_n(x) \leq \int_T e^{\delta_0 \|x\|} dA_n(x) < \int_{\mathbb{R}^n} e^{\delta_0 \|x\|} dA_n(x) < \infty.
\]

Hence, the monomial \( z^\alpha \in L^2_h(D) \). \( \square \)

**Remark 3.5.26.** Up to our knowledge, so far there is no complete description for \( b \)-complete unbounded Reinhardt domains \( ? \)
Chapter 1

\[ A_+ := \{ x \in A : x \geq 0 \} \ (A \subset \mathbb{R}), \ A_+^n := (A_+)^n, \ \text{e.g.} \ \mathbb{R}_+ = [0, +\infty), \ Z_+ = \{0, 1, 2, \ldots\}, \mathbb{R}_+^n, \mathbb{Z}_+^n \]  

N = \{1, 2, \ldots\} \ the \ set \ of \ natural \ numbers \hspace{1cm} 7

E = \ the \ unit \ disc \hspace{1cm} 7

\[ m_E(a, z) := | \frac{z - a}{1 - \overline{a}z} | \hspace{1cm} = \text{the Möbius distance} \]  

\[ p_E := \frac{1}{2} \log \frac{1 + m_E}{1 - m_E} \hspace{1cm} = \text{the Poincaré distance} \]  

\[ c_G = \text{the Carathéodory pseudodistance} \hspace{1cm} 8 \]

\[ m_G^{(k)} = \text{k-th Möbius function} \hspace{1cm} 8 \]

\[ \text{ord}_a f = \text{the order of zero of } f \text{ at } a \hspace{1cm} 8 \]

\[ g_G = \text{the pluricomplex Green function} \hspace{1cm} 8 \]

\[ \mathcal{PSH} \hspace{1cm} = \text{the family of all functions plurisubharmonic on } G \hspace{1cm} 8 \]

\[ \| \| = \text{the Euclidean norm} \hspace{1cm} 8 \]

\[ \tilde{k}_G := \text{the Lempert function} \hspace{1cm} 8 \]

\[ k_G = \text{the Kobayashi pseudodistance} \hspace{1cm} 8 \]

\[ H_G = \text{the Hahn function} \hspace{1cm} 8 \]

\[ \text{Reg } M = \text{the set of regular points} \hspace{1cm} 9 \]

\[ A_* := A \setminus \{0\} \ (A \subset \mathbb{C}^n), \ A_*^n := (A_*)^n, \ \text{e.g.} \ E_*, \mathbb{C}_*, (\mathbb{Z}_*)^n, \mathbb{C}_*^n \hspace{1cm} 9 \]

\[ \gamma_G = \text{the Carathéodory–Reiffen pseudometric} \hspace{1cm} 10 \]

\[ \gamma_G^{(k)} = \text{the k-th Reiffen pseudometric} \hspace{1cm} 10 \]

\[ A^*_G = \text{the Azukawa pseudometric} \hspace{1cm} 10 \]

\[ \omega_G = \text{the Kobayashi–Royden pseudometric} \hspace{1cm} 10 \]

\[ h_G = \text{the Hahn pseudometric} \hspace{1cm} 11 \]

\[ \mathfrak{D}(G) \hspace{1cm} 11 \]

\[ a, r := \{ z \in \mathbb{C}^n : \|z - a\| < r \}, \ B(r) := B(0, r), \mathbb{B}_a := B(1) \hspace{1cm} 11 \]

\[ L_\rho(a) \hspace{1cm} 12 \]

\[ \rho^F \hspace{1cm} 12 \]

\[ \rho^* \hspace{1cm} 12 \]

\[ \rho^\infty, \rho^\ast, \rho^\omega \hspace{1cm} 12 \]

\[ \mathcal{M}(G, \mathfrak{K}) \hspace{1cm} 12 \]

\[ j_\eta = \text{the integrated form of } \eta \hspace{1cm} 13 \]
Symbols

\[ L_q(a) \] 13
\[ h = \text{the Buseman seminorm} \] 13
\[ \bar{\eta} = \text{the Buseman pseudometric} \] 13
\[ \Delta_G = \text{the Kobayashi–Busemann pseudometric} \] 13
\[ \mathbb{D}_p = \text{the weak derivative of } \rho \] 14
\[ \mathbb{E}_p := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p} < 1 \} = \text{complex ellipsoid} \] 16
\[ A_{>0} := \{ x \in A : x > 0 \} \quad (A \subset \mathbb{R}) \quad A_{>0}^n := (A_{>0})^n, \text{ e.g. } \mathbb{R}_{>0}^n \] 17
\[ I(h) := \{ X \in \mathbb{C}^n : h(X) < 1 \} \] 19
\[ U(h) \] 19
\[ \text{Vol}(s_0) \] 20
\[ \Lambda_k = \text{the Lebesgue measure in } \mathbb{R}^k \] 20
\[ s^h \] 21
\[ \mathbb{I}_n = \text{the unit matrix} \] 21
\[ W\eta = \text{the Wu pseudometric} \] 22
\[ D_{\alpha,c} := \{ z \in \mathbb{C}^n : \text{if } \alpha_j < 0 \text{, then } z_j \neq 0, |z_1|^\alpha \cdots |z_n|^\alpha < e^c \} \quad D_{\alpha} := D_{\alpha,0} \] 27
\[ |z|^\alpha := |z_1|^\alpha \cdots |z_n|^\alpha \quad (\alpha \in \mathbb{R}^n) \] 27
\[ G_2 = \text{the symmetrized bidisc} \] 31
\[ \sigma_2 := \{ (\lambda_1 + \lambda_2, \lambda_1 \lambda_2) : \lambda_1, \lambda_2 \in \partial E \} \] 31
\[ \Sigma_2 := \{ (2\lambda^2 : \lambda \in E) \} \] 31
\[ h_{\alpha}(\lambda) := \frac{\lambda - \alpha}{1 - \alpha} \] 31
\[ F_{\alpha}(s,p) := \frac{2\alpha s \cdot p}{s^2 - np} \] 31
\[ A(m \times n) = \text{the set of all } (m \times n) \text{–matrices with entries in } A \] 42
\[ g_G(p, \cdot) = \text{the generalized Green function} \] 43
\[ |p| := \{ z \in G : p(z) > 0 \} \] 43
\[ g_G(A, \cdot) = g_G(\chi_A, \cdot) \] 43
\[ m_G(p, z) = \text{the generalized Möbius function} \] 43
\[ m_G(A, \cdot) := m_G(\chi_A, \cdot) \] 43
\[ m_G(\cdot, z) := m_G(\{ a \}, \cdot) \] 43
\[ \mathbb{R}^C := \{ p : G \rightarrow \mathbb{R}_+ \} \] 44
\[ d_G(A, \cdot) := d_G(\chi_A, \cdot) \] 44
\[ d_G(\cdot, a) := d_G(\{ a \}, \cdot) \] 44
\[ d_G^{\text{fin}}(p, \cdot) \] 44
\[ d_G^{\text{BR}}(p, \cdot) \] 44
\[ \bar{F}_e(p, \cdot) \] 44
\[ \bar{q}_p(a) := q(F(a)) \text{ ord}_a(F - F(a)) \] 47
\[ A_G(\cdot) := \{ z \in G : z_1 \cdots z_k = 0 \} \] 52
\[ \mathcal{P}(\mathbb{C}^n) = \text{the space of all complex polynomials of } n \text{–complex variables} \] 58
\[ \omega_{A,G} = \text{the relative extremal function} \] 72
\[ E_{\mathbb{X}} \] 74
\[ \mathbb{E}_{\text{Poi}} \] 74
\[ C \] 
\[ \pi \] 
\[ G \] 
\[ \log \] 
\[ 1 \] 
\[ L \] 
\[ H \] 
\[ \delta \] 
\[ \omega \] 
\[ P \] 
\[ x \] 
\[ k \] 
\[ \nu \] 
\[ \chi \] 
\[ M \] 
\[ cap \] 
\[ A \] 
\[ \alpha \] 
\[ \delta \] 
\[ \mathcal{H}^\infty \] 
\[ K \] 
\[ \mathcal{O} \] 
\[ \mathcal{A} \] 
\[ \mathcal{C} \] 
\[ \mathcal{D} \] 
\[ \mathcal{E} \] 
\[ \mathcal{F} \] 
\[ \mathcal{G} \] 
\[ \mathcal{H} \] 
\[ \mathcal{J} \] 
\[ \mathcal{K} \] 
\[ \mathcal{M} \] 
\[ \mathcal{N} \] 
\[ \mathcal{P} \] 
\[ \mathcal{Q} \] 
\[ \mathcal{R} \] 
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\[ \mathcal{T} \] 
\[ \mathcal{U} \] 
\[ \mathcal{V} \] 
\[ \mathcal{W} \] 
\[ \mathcal{X} \] 
\[ \mathcal{Y} \] 
\[ \mathcal{Z} \] 

Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>( \mathcal{E}_{\text{Pre}} )</td>
<td>the Cauchy transform of a compact set</td>
</tr>
<tr>
<td>( \mathcal{E}_{\text{P}} )</td>
<td>the equilibrium measure of a non polar compact set</td>
</tr>
<tr>
<td>( \mathcal{D}(\mathcal{G}) )</td>
<td>the minimal ball</td>
</tr>
<tr>
<td>( \mathcal{M}(\mathcal{G}) )</td>
<td>the logarithmic capacity of ( \mathcal{G} ), ( \mathcal{M}(\mathcal{G}) = { z \in \mathbb{C}^n :</td>
</tr>
<tr>
<td>( \mathcal{D}(\mathcal{G}) )</td>
<td>the space of all bounded holomorphic functions on ( \mathcal{G} ),</td>
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Chapter 2

| \( V_j \) | \( \{ z \in \mathbb{C}^n : z_j = 0 \}, \) |
| \( \log \mathcal{G} \) | \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (e^{x_1}, \ldots, e^{x_n}) \in \mathcal{G} \}, \) |
| \( \mathcal{G}(A, C) \) | quasi-elementary Reinhardt domain, |
| \( \pi_{\mathcal{G}(1, \ldots, n)}(z_1, \ldots, z_n) \) | |
| \( \mathcal{H}(\mathcal{G}) \) | the envelope of holomorphy of a domain \( \mathcal{G} \), |

Chapter 3

| \( L^2(\mathcal{G}) \) | the space of square integrable holomorphic functions on \( \mathcal{G} \), |
| \( \mathcal{K}_{\mathcal{G}}(\cdot, \cdot) \) | the Bergman kernel function on \( \mathcal{G} \), |
| \( k_{\mathcal{G}}(\cdot) \) | the Bergman kernel of \( \mathcal{G} \), |
| \( M_{\mathcal{G}} \) | the minimal ball |
| \( A_{\mathcal{G}} = A_{\mathcal{G}}(D) = \{ w \in D : \log g_{\mathcal{D}}(z, w) \leq -1 \} \) | |
| \( \nu_k \) | the equilibrium measure of a non polar compact set \( k \), |
| \( \operatorname{cap} M \) | the logarithmic capacity of \( M \), |
| \( f_{\mathcal{G}} \) | the Cauchy transform of a compact set \( \mathcal{G} \), |
| \( \alpha_{\mathcal{G}}(\cdot) \) | the potential theoretic function of \( \mathcal{G} \), |
| \( A_{\mathcal{G}}(z) \) | the annulus with center \( z \) and radii \( \frac{1}{2, \pi r}, \frac{1}{2^1, \pi r} \), |
| \( \mathcal{K}^{n}_{\mathcal{G}}(\cdot) \) | the \( n \)-th Bergman kernel |
| \( \beta_{\mathcal{G}}(z, X) \) | the Bergman pseudometric |
| \( M_{\mathcal{G}}(z, X) = \sup \{ |f'(z) X| : f \in L^2_{\mathcal{G}}(G), \| f \|_{L^2_{\mathcal{G}}(G)} = 1, f(z) = 0, z \in G \subset \mathbb{C}^n, X \in \mathbb{C}^n \} \) | |
| \( A_{\mathcal{G}}(D, r, v) = \{ z \in D : \log g_{D}(z, w) < -r \} \) | |
| \( \mathcal{G}(\Omega, a) = \mathcal{G}(\Omega) = \{ v \in \mathbb{R}^n : a + e^v \Omega \subset \Omega \}, \) |
| \( \mathcal{C}(D) = \{ v \in \mathcal{C}(\Omega_D) : \exp(a + e^v \Omega_D) \subset D \}, \) |
| \( \mathcal{C}(D) \) | where \( \Omega_D = \log D \) and \( D \) is a pseudoconvex Reinhardt domain |
| \( 1 := (1, \ldots, 1) \) | |

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